

# Cellular Stratified Spaces I: Face Categories and Classifying Spaces

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## Abstract

The notion of cellular stratified spaces was introduced in a joint work of the author with Basabe, González, and Rudyak [BGRT] with the aim of constructing a cellular model of the configuration space of a sphere. In particular, it was shown that the classifying space (order complex) of the face poset of a totally normal regular cellular stratified space  $X$  can be embedded in  $X$  as a strong deformation retract.

Here we elaborate on this idea and develop the theory of cellular stratified spaces. We introduce the notion of cylindrically normal cellular stratified spaces and associate a topological category  $C(X)$ , called the face category, to such a stratified space  $X$ . We show that the classifying space  $BC(X)$  of  $C(X)$  can be naturally embedded into  $X$ . When  $X$  is a cell complex, the embedding is a homeomorphism and we obtain an extension of the barycentric subdivision of regular cell complexes. Furthermore, when the cellular stratification on  $X$  is locally polyhedral, we show that  $BC(X)$  is a deformation retract of  $X$ .

We discuss possible applications at the end of the paper. In particular, the results in this paper can be regarded as a common framework for the Salvetti complex for the complement of a complexified hyperplane arrangement and a version of Morse theory due to Cohen, Jones, and Segal.

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# 1 Introduction

## 1.1 Motivations

Consider the following problem:

**Problem 1.1.** Given a space  $X$ , construct a combinatorial model for the configuration space  $\text{Conf}_k(X)$  of  $k$  distinct points in  $X$ . In other words, find a regular cell complex or a simplicial complex  $C_k(X)$  embedded in  $\text{Conf}_k(X)$  as a  $\Sigma_k$ -equivariant deformation retract.

Several solutions are known in special cases.

**Example 1.2.** For a finite CW-complex  $X$  of dimension 1, namely a graph, Abrams constructed a subspace  $C_k^{\text{Abrams}}(X)$  contained in  $\text{Conf}_k(X)$  in his thesis [Abr00] and proved that there is a homotopy equivalence

$$C_k^{\text{Abrams}}(X) \simeq \text{Conf}_k(X)$$

as long as the following two conditions are satisfied:

1. each path connecting vertices in  $X$  of valency more than 2 has length at least  $k + 1$ , and
2. each homotopically essential path connecting a vertex to itself has length at least  $k + 1$ .

Here a path means a 1-dimensional subcomplex homeomorphic to a closed interval.  $\square$

**Example 1.3.** Consider the case  $X = \mathbb{R}^n$ . For  $1 \leq i < j \leq k$ , the hyperplane

$$H_{i,j} = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_i = x_j\}$$

in  $\mathbb{R}^k$  defines a linear subspace  $H_{i,j} \otimes \mathbb{R}^n$  in  $\mathbb{R}^k \otimes \mathbb{R}^n = \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_k = X^k$  and we have

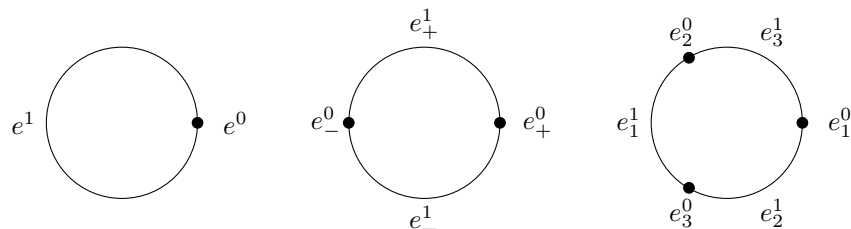
$$\text{Conf}_k(\mathbb{R}^n) = X^k - \bigcup_{1 \leq i < j \leq k} H_{i,j} \otimes \mathbb{R}^n.$$

The collection  $\{H_{i,j} \mid 1 \leq i < j \leq k\}$  is called the braid arrangement of rank  $k - 1$  and is denoted by  $\mathcal{A}_{k-1}$ .

When  $n = 2$ , the construction due to Salvetti [Sal87] gives us a regular cell complex  $\text{Sal}(\mathcal{A}_{k-1})$  embedded in  $\text{Conf}_k(\mathbb{R}^2)$  as a  $\Sigma_k$ -equivariant deformation retract. More generally, the construction sketched at the end of [BZ92] by Björner and Ziegler and elaborated in [DCS00] by De Concini and Salvetti gives us a regular cell complex  $\text{Sal}^{(n)}(\mathcal{A}_{k-1})$  embedded in  $\text{Conf}_k(\mathbb{R}^n)$  as a  $\Sigma_k$ -equivariant deformation retract.

This construction is a special case of the construction of a regular cell complex whose homotopy type represents the complement of the subspace arrangement associated with a real hyperplane arrangement.  $\square$

There are pros and cons in these two constructions. The conditions in Abrams' theorem require us to subdivide a given 1-dimensional CW-complex finely. For example, his construction fails to give the right homotopy type of the configuration space  $\text{Conf}_2(S^1)$  of two points in  $S^1$  when it is applied to the minimal cell decomposition;  $S^1 = e^0 \cup e^1$ . The minimal regular cell decomposition  $S^1 = e_-^0 \cup e_+^0 \cup e_-^1 \cup e_+^1$  is not fine enough, either. We need to subdivide  $S^1$  into three 1-cells to use Abrams' model.



Another problem is that his theorem is restricted to 1-dimensional CW-complexes, although the construction of the model itself works for any cell complex<sup>1</sup>.

The second construction suggests that we should consider more general stratifications than cell decompositions. The complex  $\text{Sal}^{(n)}(\mathcal{A}_{k-1})$  is constructed from the combinatorial structure of the “cell decomposition” of  $\mathbb{R}^k \otimes \mathbb{R}^n$  defined by the hyperplanes in the arrangement  $\mathcal{A}_{k-1}$  together with the standard framing in  $\mathbb{R}^n$ . “Cells” in this decomposition are unbounded regions in  $\mathbb{R}^k \otimes \mathbb{R}^n$ . Although such a decomposition is not regarded as a cell decomposition in the usual sense, we may extend the definition of face posets to such generalized cell decompositions. The crucial deficiency of the second construction is, however, that it works only for Euclidean spaces.

One of the motivations for this paper is to find a common framework for working with configuration spaces and complements of arrangements. Although there are many interesting “parallel theories” between configuration spaces and arrangements, e.g. the Fulton-MacPherson-Axelrod-Singer compactification [FM94, AS94] and the De Concini-Procesi wonderful model [DCP95], there is no “Salvetti complex” for configuration spaces in general. A more concrete motivation is, therefore, to solve Problem 1.1 in such a way that it generalizes the Salvetti complex for the braid arrangement.

By analyzing the techniques of combinatorial algebraic topology used in the proof of Salvetti’s theorem, the notion of cellular stratified spaces was introduced in [BGRT]. We also introduced the notion of totally normal cellular stratified spaces to handle characteristic maps combinatorially.

**Definition 1.4** (Definition 3.7). Let  $X$  be a normal<sup>2</sup> cellular stratified space.  $X$  is called *totally normal* if, for each  $n$ -cell  $e_\lambda$  with characteristic map  $\varphi_\lambda : D_\lambda \rightarrow X$ ,

<sup>1</sup>It seems the study of higher dimensional cases has just started [AGH].

<sup>2</sup>Definition 2.7

1. there exists a structure of regular cell complex on  $S^{n-1}$  containing  $\partial D_\lambda$  as a stratified subspace, and
2. for any cell  $e$  in  $\partial D_\lambda$ , there exists a cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the characteristic map of  $e_\mu$  factors through  $D_\lambda$  via the characteristic map of  $e$ :

$$\begin{array}{ccccccc}
 \bar{e} & \hookrightarrow & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & X \\
 \uparrow & & & & & \nearrow \varphi_\mu & \\
 D & \xlongequal{\quad} & D_\mu & & & & 
 \end{array}$$

The following result says that we can always recover the homotopy type of  $X$  from its face poset, if  $X$  is totally normal and regular.

**Theorem 1.5** (Theorem 6.2.4 in [BGRT]). *For a totally normal regular CW cellular stratified space  $X$ , the classifying space  $BF(X)$  of the face poset  $F(X)$  can be embedded in  $X$  as a strong deformation retract.*

When  $X = \mathbb{R}^k \otimes \mathbb{R}^n$  and the stratification is given by a real hyperplane arrangement  $\mathcal{A}$  in  $\mathbb{R}^k$  by the method of Björner-Ziegler and De Concini-Salvetti [BZ92, DCS00], then  $BF(X)$  is homeomorphic to the higher order Salvetti complex  $\text{Sal}^{(n)}(\mathcal{A})$ .

Note that, if  $X$  is a regular cell complex, it is a fundamental fact in combinatorial algebraic topology that the classifying space  $BF(X)$  is homeomorphic to  $X$ . The above result is a generalization of this well-known fact. In [BGRT], a combinatorial model for  $\text{Conf}_k(S^n)$  was constructed by defining a totally normal regular cellular stratification on  $\text{Conf}_k(S^n)$  by using the braid arrangements and then by applying the above theorem. Consequently the homotopy dimension of  $\text{Conf}_k(S^n)/\Sigma_k$  was determined in [BGRT], resulting in a good estimate of the higher symmetric topological complexity of  $S^n$ .

Examples of totally normal cellular stratified spaces include:

- regular CW complexes,
- the Björner-Ziegler stratification [BZ92] of Euclidean spaces defined by subspace arrangements,
- graphs regarded as 1-dimensional cell complexes,
- the minimal cell decomposition of  $\mathbb{R}P^n$ ,
- Kirillov's PLCW-complexes [Kir] satisfying a certain regularity condition, and
- the geometric realization of  $\Delta$ -sets.

By studying these examples, the author noticed that the regularity assumption in Theorem 1.5 can be removed by modifying the definition of the face poset. We should associate an acyclic category to a totally normal cellular stratified space, instead of a poset.

Another source of inspirations for this paper is a preprint [CJS] of R. Cohen, J.D.S. Jones, and G.B. Segal. Given a Morse-Smale bowl function  $f : M \rightarrow \mathbb{R}$  on a smooth closed manifold  $M$ , they proposed a construction of an acyclic topological category  $C(f)$  by using critical points and moduli spaces of flow lines. They stated that the classifying space  $BC(f)$  is homeomorphic to  $M$ . Their observation strongly suggests that a “acyclic topological category version” of Theorem 1.5 should exist.

## 1.2 Statements of Results

The aim of this project is to develop the theory of cellular stratified spaces by extending these results and ideas. For this purpose, we first introduce the notion of cylindrical structures on cellular stratified spaces.

**Definition 1.6** (Definition 3.21). A *cylindrical structure* on a normal cellular stratified space  $X$  consists of

- a normal stratification on  $S^{n-1}$  containing  $\partial D_\lambda$  as a stratified subspace for each  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  in  $X$ ,
- a stratified space  $P_{\mu,\lambda}$  and a morphism of stratified spaces  $b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \rightarrow \partial D_\lambda$  for each pair of cells  $e_\mu \subset \partial e_\lambda$ , and
- a morphism of stratified spaces  $c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \rightarrow P_{\lambda_0,\lambda_2}$  for each sequence  $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$ ,

satisfying certain compatibility and associativity conditions. A cellular stratified space equipped with a cylindrical structure is called a *cylindrically normal* cellular stratified space.

Examples of cylindrically normal cellular stratified spaces include:

- totally normal cellular stratified spaces,
- PLCW complexes,
- the minimal cell decomposition of  $\mathbb{CP}^n$ , and
- the geometric realization of simplicial sets.

The cell decomposition on a smooth manifold induced by a Morse-Smale bowl function seems to give us another good example. Cohen, Jones, and Segal [CJS] constructed a structure very close to a cylindrical structure by using the “moduli spaces of flows” of a Morse-Smale bowl function.

In the case of the function  $f([z_0, z_1, z_2]) = |z_0|^2 + 2|z_2|^2 + 3|z_3|^2$  on  $\mathbb{CP}^2 = S^5/S^1$ , the structure is explicitly described in §5 of their preprint<sup>3</sup>. Alternatively and more generally, we can construct the same cylindrical structure by identifying  $\mathbb{CP}^n$  with the Davis-Januszkiewicz construction  $M(\lambda_n)$ <sup>4</sup>. This observation suggests that a large class of quasitoric and torus manifolds have cylindrically normal cell decompositions induced from certain stratifications on the associated simple polytopes.

Given a cylindrically normal cellular stratified space  $X$ , we define an acyclic topological category  $C(X)$ , called the cylindrical face category of  $X$ . Objects are cells in  $X$  and the space of morphisms from  $e_\mu$  to  $e_\lambda$  is defined to be  $P_{\mu,\lambda}$ .

Our first result says that the classifying space  $BC(X)$  of  $C(X)$  can be always embedded in  $X$ .

**Theorem 1.7** (Theorem 4.15). *For any cylindrically normal CW cellular stratified space  $X$ , there exists an embedding*

$$\tilde{i} : BC(X) \hookrightarrow X$$

*which is natural with respect to morphisms<sup>5</sup> of cylindrically normal cellular stratified spaces.*

*Furthermore, when all cells in  $X$  are closed,  $\tilde{i}$  is a homeomorphism.*

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<sup>3</sup>Example 3.26

<sup>4</sup>Example 3.27

<sup>5</sup>Definition 3.32

When  $X$  contains non-closed cells,  $\tilde{i}$  is not a homeomorphism. Our second result says that, under a reasonable condition, those non-closed cells can be collapsed into  $BC(X)$ .

**Theorem 1.8** (Theorem 4.16). *For a locally polyhedral<sup>6</sup> cellular stratified space  $X$ , the image of the embedding  $\tilde{i} : BC(X) \hookrightarrow X$  is a strong deformation retract of  $X$ . The deformation retraction can be taken to be natural with respect to morphisms of locally polyhedral cellular stratified spaces.*

These two results generalize Theorem 1.5 (Theorem 6.2.8 in [BGRT]). It turns out that Theorem 1.7 still holds when we replace cells by “star-shaped” cells. We introduce the notion of stellar stratified spaces and prove Theorem 1.7 for CW cylindrically normal stellar stratified spaces. We investigate more details of stellar structures in a separate paper [Tama], in which we show that the functor  $BC(-)$  transforms cellular stratified spaces to stellar stratified spaces in a canonical way.

Examples of cylindrical structures suggest that we may apply these theorems to the following problems:

1. Construct combinatorial models for configuration spaces and apply them to study the homotopy types of configuration spaces.
2. Develop a refinement and an extension of the Cohen-Jones-Segal Morse theory.

We should also be able to apply the results in this paper to other types of configurations and arrangements. We might be able to apply the results to toric topology. The usefulness of Forman’s discrete Morse theory [For95, For98] for posets in combinatorial algebraic topology suggests that an extension of Forman’s theory to acyclic topological categories will become a good tool to study cellular stratified spaces. We briefly discuss these problems and possible applications at the end of this paper.

We need various operations on cellular stratified spaces, such as products and subdivisions, in order to apply the results developed here to configuration spaces. We postpone discussions on these topics to the second part [Tama] of this series of papers.

### 1.3 Organization

The paper is organized as follows.

- §2 is preliminary. We fix notation and terminology for stratified spaces in §2.1. The definition of cell structures is recalled from [BGRT] in §2.2. Our definition of cellular stratified spaces in §2.3 is slightly different from the one in [BGRT] but essentially the same. We introduce stellar stratified spaces in §2.4.
- “Niceness conditions” for cellular and stellar stratified spaces are discussed in §3. We recall several variations of normality and regularity conditions from [BGRT] in §3.1. A new structure, called cylindrically normal stellar stratification, is introduced in §3.2. Finally we introduce locally polyhedral structures in §3.3.
- We state and prove the main results in §4. We define two kinds of face categories for a certain class of cellular and stellar stratified spaces in §4.1. Finally in §4.2, we construct an embedding of the classifying space of the face category of a cylindrically normal stellar stratified space. We prove that the image of the embedding is a strong deformation retract for locally polyhedral cellular stratified spaces.

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<sup>6</sup>Definition 3.34

- We conclude this paper by making a couple of remarks in §5.

The paper contains two appendices for the convenience of the reader.

- In Appendix A, we recall definitions and properties of simplicial complexes, simplicial sets, and related structures.
- Appendix B is a summary on basics of topological categories, including their classifying spaces.

## Acknowledgments

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The results we obtained on the Google Wave have been incorporated in a joint work [BGRT] with Ibai Basabe and Yuli Rudyak. The current work is an attempt to extend the last two sections of that paper. I am grateful to my coauthors, as well as Peter Landweber, for discussions on cellular stratified spaces. I would also like to thank those who send me comments on early drafts of this paper, especially Mikiya Masuda, Takashi Mukouyama, Peter Landweber, Jesús González, and Kouyemon Iriye.

Before I began the collaboration with González on the Google Wave, some of the ideas had already been developed during the discussions with my students. Takamitsu Jinno and Mizuki Furuse worked on Hom complexes in 2009 and configuration spaces of graphs in 2010, respectively, in their master’s theses. The possibility of finding a better combinatorial model for configuration spaces was suggested by their work. The connection with Cohen-Jones-Segal Morse theory, which resulted in the current definition of cylindrical structure, was discovered during discussions with another student, Kohei Tanaka. The relation between cylindrical structures on cellular stratified spaces and Cohen-Jones-Segal Morse theory is currently being investigated by Tanaka. An application of cellular stratified spaces to configuration spaces of graphs was announced during the 2011 International Symposium on Nonlinear Theory and its Applications (NOLTA2011) and details will appear soon as a joint work with Furuse [FT].

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## 2 Stratifications and Cells

This section is preliminary. We introduce notions of stratified spaces, cell structures, and cellular stratified spaces. Although the term “stratified space” has been used in singularity theory since its beginning and is a well-established concept, we need our own definition of stratified spaces.

### 2.1 Stratified Spaces

Before we introduce *cellular* stratified spaces, let us first recall the notion of stratified spaces in general, whose theory has been developed in singularity theory. Unfortunately, however, there seems to be no standard definition of stratified spaces. There are many non-equivalent definitions in the literature. For this reason, we decided to examine several books and extract properties

for our needs. As our prototypes, we use definitions in books by Kirwan [Kir88], Bridson and Haefliger [BH99], Pflaum [Pfl01], and Schürmann [Sch03].

Here is our reformulation.

**Definition 2.1.** Let  $X$  be a topological space and  $\Lambda$  be a poset. A *stratification* of  $X$  indexed by  $\Lambda$  is a map

$$\pi : X \longrightarrow \Lambda$$

satisfying the following properties:

1. For  $\lambda \in \text{Im } \pi$ ,  $\pi^{-1}(\lambda)$  is connected and locally closed<sup>7</sup>.
2. For  $\lambda, \lambda' \in \text{Im } \pi$ ,  $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\lambda')}$  if and only if  $\lambda \leq \lambda'$ .

For simplicity, we put  $e_\lambda = \pi^{-1}(\lambda)$  and call it a *stratum* with index  $\lambda$ .

**Remark 2.2.** We may safely assume that  $\pi$  is surjective. When we define morphisms of stratified spaces and stratified subspaces, however, it is more convenient not to assume the surjectivity.

**Remark 2.3.** The second condition in the definition of stratification is equivalent to saying that  $\pi$  is continuous when  $\Lambda$  is equipped with the Alexandroff topology. Recall that the Alexandroff topology on a set  $\Lambda$  associated with a preorder  $\leq$  is the topology in which closed sets are given by those subsets  $D$  satisfying the condition that, for any  $\mu, \lambda \in \Lambda$ ,  $\lambda \in D$  and  $\mu \leq \lambda$  imply  $\mu \in D$ .

In other words, a stratification on a topological space  $X$  is a grading defined by a poset in the monoidal category of topological spaces. See [Tamb] for more details on gradings by a comonoid object in a monoidal category.

**Remark 2.4.** In Definition 2.1, we may safely assume that  $\Lambda$  is a poset, instead of a preordered set, because of our local-closedness assumption on each stratum.

Given a map  $\pi : X \rightarrow \Lambda$ , we have a decomposition of  $X$ , i.e.

1.  $X = \bigcup_{\lambda \in \text{Im } \pi} e_\lambda$ .
2. For  $\lambda, \lambda' \in \text{Im } \pi$ ,  $e_\lambda \cap e_{\lambda'} = \emptyset$  if  $\lambda \neq \lambda'$ .

Thus the image of  $\pi$  in the indexing poset  $\Lambda$  can be identified with the set of strata. This observation justifies the following terminology.

**Definition 2.5.** For a stratification  $\pi : X \rightarrow \Lambda$ , the image  $\text{Im } \pi$  is called the *face poset* and is denoted by  $P(X, \pi)$  or simply by  $P(X)$ .

When  $P(X)$  is finite or countable,  $(X, \pi)$  is said to be *finite* or *countable*.

**Remark 2.6.** The above structure (without the connectivity of  $\pi^{-1}(\lambda)$ ) is called a decomposition in Pflaum's book [Pfl01]. Pflaum used the notion of set germ to define a stratification from local decompositions. Furthermore, Pflaum imposed three further conditions:

- If  $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ , then  $e_\mu \subset \overline{e_\lambda}$ .
- Each stratum is a smooth manifold.
- The collection  $\{e_\lambda\}_{\lambda \in \Lambda}$  is locally finite in the sense that, for any  $x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap e_\lambda \neq \emptyset$  only for a finite number of strata  $e_\lambda$ .

---

<sup>7</sup>A subset  $A$  of a topological space  $X$  is said to be *locally closed*, if every point  $x \in A$  has a neighborhood  $U$  in  $X$  with  $A \cap U$  closed in  $U$ . This condition is known to be equivalent to saying that  $A$  is an intersection of an open and a closed subset of  $X$ , or  $A$  is open in its closure  $\overline{A}$ .



The first condition corresponds to the normality of cell complexes. We would like to separate the third condition as one of the conditions for CW stratifications. For the second condition, as is remarked in his book, we may replace smooth manifolds by any collection of geometric objects such as complex manifolds, real analytic sets, polytopes, and so on. In §2.3, we choose the class of spaces equipped with “cell structures” and define the notion of cellular stratified spaces. As an example of another choice, we use “star-shaped cells” and define the notion of stellar stratified spaces in §2.4, which will be used to extend dualities of simplicial complexes to cellular stratified spaces. We also impose more structures on cells and introduce notions of locally polyhedral structures in Definition 3.34.

Bridson and Haefliger [BH99] define stratifications by using closed strata. Their strata correspond to closures of strata in our definition. Furthermore, they also required their stratifications to be normal in the following sense.

**Definition 2.7.** We say a stratum  $e_\lambda$  in a stratified space  $(X, \pi)$  is *normal* if  $e_\mu \subset \overline{e_\lambda}$  whenever  $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ . When all strata are normal, the stratification  $\pi$  is said to be *normal*.

It is immediate to verify the following.

**Lemma 2.8.** *A stratum  $e_\lambda$  is normal if and only if  $\partial e_\lambda = \overline{e_\lambda} - e_\lambda$  is a union of strata.*

Another difference between our definition and the one given by Bridson and Haefliger is that they considered intersections of closed strata.

**Lemma 2.9.** *Let  $(X, \pi)$  be a normal stratified space. Then, for any pair of strata  $e_\mu, e_\lambda$ , the intersection  $\overline{e_\mu} \cap \overline{e_\lambda}$  is a union of strata.*

*Proof.* This is obvious, since closures of different strata can intersect only on the boundaries, which are unions of strata by the definition of normality.  $\square$

The following is a typical example of stratifications we are interested in.

**Example 2.10.** Let  $S_1 = \{-1, 0, 1\}$  with poset structure  $0 < \pm 1$ . The sign function

$$\text{sign} : \mathbb{R} \longrightarrow S_1$$

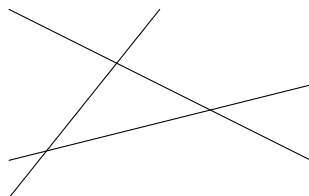
given by

$$\text{sign}(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

defines a stratification on  $\mathbb{R}$ :

$$\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty).$$

This innocent-looking stratification turns out to be one of the most important ingredients in the theory of real hyperplane arrangements. Let  $\mathcal{A} = \{H_1, \dots, H_k\}$  be a real affine hyperplane arrangement in  $\mathbb{R}^n$  defined by affine 1-forms  $L = \{\ell_1, \dots, \ell_k\}$ . Hyperplanes cut  $\mathbb{R}^n$  into convex regions that are homeomorphic to the interior of the  $n$ -disk. Each hyperplane  $H_i$  is cut into convex regions of dimension  $n - 1$  by other hyperplanes, and so on.



These cuttings can be described as a stratification defined by the sign function as follows. Define a map

$$\pi_{\mathcal{A}} : \mathbb{R}^n \longrightarrow \text{Map}(L, S_1)$$

by

$$\pi_{\mathcal{A}}(a)(\ell_i) = \text{sign}(\ell_i(a)).$$

The partial order in  $S_1$  induces a partial order on  $\text{Map}(L, S_1)$  by  $\varphi \leq \psi$  if and only if  $\varphi(\ell_i) \leq \psi(\ell_i)$  for all  $i$ . Then  $\pi_{\mathcal{A}}$  is the indexing map for the stratification of  $\mathbb{R}^n$  induced by  $\mathcal{A}$ .

There is a standard way to extend the above construction to a stratification on  $\mathbb{C}^n$  by the complexification  $\mathcal{A} \otimes \mathbb{C} = \{H_1 \otimes \mathbb{C}, \dots, H_k \otimes \mathbb{C}\}$  as is studied intensively in the theory of hyperplane arrangements. Here the complexification  $H \otimes \mathbb{C}$  of a hyperplane  $H$  in  $\mathbb{R}^n$  is the complex hyperplane in  $\mathbb{C}^n$  given by regarding the defining equation for  $H$  as an equation in  $\mathbb{C}^n$ . A good reference for such a stratification is the paper [BZ92] by Björner and Ziegler. As is sketched at the end of the above paper, the stratification of  $\mathbb{R}^n$  defined above can be extended to a stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  as follows: Let  $\mathcal{A} = \{H_1, \dots, H_k\}$  and  $L = \{\ell_1, \dots, \ell_k\}$  be as above. For each  $i$  define

$$\ell_i \otimes \mathbb{R}^\ell : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

by  $(\ell_i \otimes \mathbb{R}^\ell)(x_1, \dots, x_\ell) = (\ell_i(x_1), \dots, \ell_i(x_\ell))$  under the identification

$$\mathbb{R}^n \otimes \mathbb{R}^\ell = \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_\ell = (\mathbb{R}^n)^\ell$$

Then maps  $\ell_1 \otimes \mathbb{R}^\ell, \dots, \ell_k \otimes \mathbb{R}^\ell$  define a subspace arrangement  $\mathcal{A} \otimes \mathbb{R}^\ell = \{H_1 \otimes \mathbb{R}^\ell, \dots, H_k \otimes \mathbb{R}^\ell\}$  in  $\mathbb{R}^n \otimes \mathbb{R}^\ell$ .

Let  $S_\ell = \{0, \pm e_1, \dots, \pm e_\ell\}$  be the poset with partial ordering  $0 < \pm e_1 < \dots < \pm e_\ell$ . Define the  $\ell$ -th order sign function

$$\text{sign}_\ell : \mathbb{R}^\ell \longrightarrow S_\ell$$

by

$$\text{sign}_\ell(\mathbf{x}) = \begin{cases} \text{sign}(x_\ell)e_\ell, & x_\ell \neq 0 \\ \text{sign}(x_{\ell-1})e_{\ell-1}, & x_\ell = 0, x_{\ell-1} \neq 0 \\ \vdots \\ \text{sign}(x_1)e_1, & x_n = \dots = x_2 = 0, x_1 \neq 0 \\ 0 & \mathbf{x} = 0. \end{cases}$$

Define a stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell} : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \text{Map}(L, S_\ell)$$

by

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}(\mathbf{x})(\ell_i) = \text{sign}_\ell((\ell_i \otimes \mathbb{R}^\ell)(\mathbf{x})).$$

This is a normal stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$ . □

**Example 2.11.** Consider the standard  $n$ -simplex

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \dots + x_n = 1, x_i \geq 0\}.$$

Define

$$\pi_n : \Delta^n \longrightarrow 2^{[n]}$$

by

$$\pi_n(x_0, \dots, x_n) = \{i \in [n] \mid x_i \neq 0\},$$

where  $[n] = \{0, \dots, n\}$  and  $2^{[n]}$  is the power set of  $[n]$ . Under the standard poset structure on  $2^{[n]}$ ,  $\pi$  is a stratification with strata simplices in  $\Delta^n$ . This is a normal stratification.

Define another stratification  $\pi_n^{\max} : \Delta^n \rightarrow [n]$  by

$$\pi_n^{\max}(x_0, \dots, x_n) = \max \{i \mid x_i \neq 0\},$$

where  $[n]$  is equipped with the partial order  $0 < 1 < \dots < n$ . The resulting decomposition is

$$\Delta^n = (\Delta^n - \Delta^{n-1}) \cup (\Delta^{n-1} - \Delta^{n-2}) \cup \dots \cup (\Delta^1 - \Delta^0) \cup \Delta^0.$$

This is also a normal stratification.  $\square$

**Example 2.12.** Let  $G$  be a compact Lie group acting smoothly on a smooth manifold  $M$ . M. Davis [Dav78] defined a stratification on  $M$  and on the quotient space  $M/G$  as follows. The indexing set  $I(G)$  is called the set of normal orbit types and is defined by

$$I(G) = \{(\varphi : H \rightarrow GL(V)) \mid H < G \text{ a closed subgroup, } \varphi \text{ a representation, } V^H = \{0\}\} / \sim,$$

where  $(\varphi : H \rightarrow GL(V)) \sim (\varphi' : H' \rightarrow GL(V'))$  if and only if there exist an element  $g \in G$  and a linear isomorphism  $f : V \rightarrow V'$  such that  $H' = gHg^{-1}$  and the diagram

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & GL(V) \\ g(-)g^{-1} \downarrow & & \downarrow f(-)f^{-1} \\ H' & \xrightarrow{\varphi'} & GL(V') \end{array}$$

is commutative.

He defined a map

$$\pi_M : M \longrightarrow I(G)$$

by

$$\pi_M(x) = (G_x, S_x / (S_x)^{G_x}),$$

where  $G_x$  is the isotropy subgroup at  $x$  and  $S_x = T_x M / T_x(Gx)$  is the normal space at  $x$  to the orbit  $Gx$ . There is a canonical partial order on  $I(G)$  and it is proved that  $\pi_M$  is a normal stratification (Theorem 1.6 in [Dav78]). It is also proved that the stratification descends to one of  $M/G$ .

In particular, for a torus manifold<sup>8</sup> or a small cover<sup>9</sup>  $M$  of dimension  $2n$  or  $n$ , the quotient  $M/T^n$  or  $M/\mathbb{Z}_2^n$  has a canonical normal stratification, respectively.

When  $M = \mathbb{CP}^n$  or  $\mathbb{RP}^n$ , the quotient is  $\Delta^n$  and the stratification by normal orbit types corresponds to the stratification  $\pi_n$  in Example 2.11. We will see the other stratification  $\pi_n^{\max}$  in Example 2.11 corresponds to the minimal cell decompositions of  $\mathbb{RP}^n$  and  $\mathbb{CP}^n$  in Example 3.27.  $\square$

**Example 2.13.** The stratification on the quotient  $M/G$  has been generalized to the notion of manifolds with faces or more generally manifolds with corners. See §6 of Davis' paper [Dav83], for example. These are also important examples of normal stratifications.  $\square$

<sup>8</sup>in the sense of Hattori and Masuda [HM03]

<sup>9</sup>in the sense of Davis and Januszkiewicz [DJ91]

It is easy to verify that the product of two stratifications is again a stratification.

**Lemma 2.14.** *Let  $(X, \pi_X)$  and  $(Y, \pi_Y)$  be stratified spaces. The map*

$$\pi_X \times \pi_Y : X \times Y \longrightarrow P(X) \times P(Y)$$

*defines a stratification on  $X \times Y$ .*

The following requirements for morphisms of stratified spaces should be reasonable.

**Definition 2.15.** Let  $(X, \pi_X)$  and  $(Y, \pi_Y)$  be stratified spaces.

- A *morphism of stratified spaces* is a pair  $\mathbf{f} = (f, \underline{f})$  of a continuous map  $f : X \rightarrow Y$  and a map of posets  $\underline{f} : P(X) \rightarrow P(Y)$  making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ P(X) & \xrightarrow{\underline{f}} & P(Y). \end{array}$$

- When  $X = Y$  and  $f$  is the identity,  $\mathbf{f} = (f, \underline{f})$  is called a *subdivision*. We also say that  $(X, \pi_X)$  is a subdivision of  $(Y, \pi_Y)$  or  $(Y, \pi_Y)$  is a *coarsening* of  $(X, \pi_X)$ .
- When  $f(e_\lambda) = e_{\underline{f}(\lambda)}$  for each  $\lambda$ , it is called a *strict morphism*.
- When  $\mathbf{f} = (f, \underline{f})$  is a strict morphism of stratified spaces and  $f$  is an embedding of topological spaces,  $\mathbf{f}$  is said to be an *embedding* of  $X$  into  $Y$ .

For a stratified space  $\pi : Y \rightarrow \Lambda$  and a continuous map  $f : X \rightarrow Y$ , the composition  $f^*(\pi) = \pi \circ f : X \rightarrow \Lambda$  may or may not be a stratification.

**Definition 2.16.** Let  $\mathbf{f} : (X, \pi_X) \rightarrow (Y, \pi_Y)$  be a morphism of stratified spaces. When  $\underline{f} : P(X) \rightarrow \pi_Y(f(X))$  is an isomorphism of posets

$$\begin{array}{ccccc} X & \xrightarrow{f} & f(X) & \hookrightarrow & Y \\ \pi_X \downarrow & & \downarrow \pi_Y & & \downarrow \pi_Y \\ P(X) & \xrightarrow{\underline{f}} & \pi_Y(f(X)) & \hookrightarrow & P(Y), \end{array}$$

we say  $\pi_X$  is *induced* from  $\pi_Y$  via  $f$ . We sometimes denote it by  $f^*(\pi_Y)$ .

**Example 2.17.** Consider the double covering map

$$2 : S^1 \longrightarrow S^1.$$

The minimal cell decomposition  $\pi_{\min} : S^1 = e^0 \cup e^1$  on  $S^1$  in the range does not induce a stratification on  $S^1$  in the domain, since the inverse images of strata are not connected. But we have a strict morphism of stratified spaces

$$\begin{array}{ccc} S^1 & \xrightarrow{2} & S^1 \\ \downarrow & & \downarrow \\ \{0_+, 0_-, 1_+, 1_-\} & \longrightarrow & \{0, 1\} \end{array}$$

if  $S^1$  in the domain is equipped with the minimal  $\Sigma_2$ -equivariant cell decomposition  $S^1 = e_-^0 \cup e_+^0 \cup e_-^1 \cup e_+^1$ .

Consider the complex analogue of the double covering on  $S^1$ , i.e. the Hopf bundle

$$\eta : S^3 \longrightarrow S^2.$$

This is a principal fiber bundle with fiber  $S^1$ . The minimal cell decomposition on  $S^2$ ,  $S^2 = e^0 \cup e^2$ , is a stratification. This stratification induces a stratification on  $S^3$

$$S^3 = \eta^{-1}(e^0) \amalg \eta^{-1}(e^2).$$

Note that we have

$$\begin{aligned} \eta^{-1}(e^0) &\cong e^0 \times S^1 \\ \eta^{-1}(e^2) &\cong e^2 \times S^1 \end{aligned}$$

The face posets of these stratifications are isomorphic to the poset  $\{0 < 2\}$  and we have a commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\eta} & S^2 \\ \downarrow & & \downarrow \\ \{0, 2\} & \equiv & \{0, 2\}. \end{array}$$

Note that these two examples can be described as

$$\begin{aligned} S^1 &= e^0 \times S^0 \amalg e^1 \times S^0 \\ S^3 &= e^0 \times S^1 \amalg e^2 \times S^1, \end{aligned}$$

suggesting the existence of a common framework for handling them simultaneously. We propose the notion of cylindrical structures as such in §3.2.  $\square$

As a special case of embeddings of stratified spaces, we have the notion of stratified subspaces.

**Definition 2.18.** Let  $(X, \pi)$  be a stratified space and  $A$  be a subspace of  $X$ . If the restriction  $\pi|_A$  is a stratification,  $(A, \pi|_A)$  is called a *stratified subspace* of  $(X, \pi)$ .

When the inclusion  $i : A \hookrightarrow X$  is a strict morphism,  $A$  is called a *strict stratified subspace* of  $X$ .

As is the case of cell complexes, the CW condition is useful.

**Definition 2.19.** A stratification  $\pi$  on  $X$  is said to be *CW* if it satisfies the following two conditions:

1. (Closure Finite) For each stratum  $e_\lambda$ ,  $\partial e_\lambda$  is covered by a finite number of strata.
2. (Weak Topology)  $X$  has the weak topology determined by the covering  $\{\overline{e_\lambda} \mid \lambda \in P(X)\}$ .

As is the case of cell complexes, finiteness implies the CW condition.

**Lemma 2.20.** Any finite stratified space is CW. Thus any finite cellular or stellar stratified space is CW.

*Proof.* When  $X = e_{\lambda_1} \cup \dots \cup e_{\lambda_n}$ ,  $X$  has the weak topology determined by the covering  $X = \overline{e_{\lambda_1}} \cup \dots \cup \overline{e_{\lambda_n}}$ . And the result follows from the following well-known fact.  $\square$

**Lemma 2.21.** *Let  $X$  be a topological space and  $\mathcal{X} = \{X_\lambda\}_{\lambda \in \Lambda}$  be a locally finite closed covering<sup>10</sup> of  $X$ . Then  $X$  has the weak topology with respect to  $\mathcal{X}$ .*

Recall that a cell complex  $X$  is called locally finite if every point  $x \in X$  has a neighborhood contained in a finite subcomplex. Since stratified subspaces are not necessarily closed, we need to impose the closedness condition in the definition of locally-finiteness for stratified spaces.

**Definition 2.22.** We say a stratified space  $(X, \pi)$  is *locally finite*, if, for any  $x \in X$ , there exists a closed finite stratified subspace  $A$  of  $X$  such that  $x \in \text{Int}(A)$ .

**Lemma 2.23.** *If  $(X, \pi)$  is a locally finite stratified space, the covering  $\{\overline{e_\lambda}\}_{\lambda \in P(X)}$  is a locally finite closed covering of  $X$ .*

*Proof.* Let  $(X, \pi)$  be a locally finite stratified space. For  $x \in X$ , by assumption, there exists a closed finite stratified subspace  $A$  of  $X$  with  $x \in \text{Int}(A)$ . The closure of each cell in  $A$  is covered by a finite number of cells. Since  $A$  is finite,  $\text{Int}(A)$  intersects with only a finite number of cells. Thus  $\{\overline{e_\lambda}\}_{\lambda \in P(X)}$  is a locally finite closed covering.  $\square$

As an immediate corollary, we obtain the following enhancement of Lemma 2.20.

**Corollary 2.24.** *Any locally finite stratified space is CW.*

## 2.2 Cells

We would like to define a cellular stratification on a topological space as a stratification on  $X$  whose strata are “cells”. As we have seen in Example 2.10, we would like to regard chambers and faces of a real hyperplane arrangement as “cells”, suggesting the need of non-closed cells.

**Definition 2.25.** A *globular  $n$ -cell* is a subset  $D$  of  $D^n$  containing  $\text{Int}(D^n)$ . We call  $D \cap \partial D^n$  the *boundary* of  $D$  and denote it by  $\partial D$ . The number  $n$  is called the *globular dimension* of  $D$ .

**Remark 2.26.** We introduce another dimension, called stellar dimension, for a more general class of subsets of  $D^n$  in §2.4.

We use the following definition of cell structures introduced in [BGRT].

**Definition 2.27.** Let  $X$  be a topological space. For a non-negative integer  $n$ , an  *$n$ -cell structure* on a subspace  $e \subset X$  is a pair  $(D, \varphi)$  of a globular  $n$ -cell  $D$  and a continuous map

$$\varphi : D \longrightarrow X$$

satisfying the following conditions:

1.  $\varphi(D) = \overline{e}$  and the restriction  $\varphi|_{\text{Int}(D^n)} : \text{Int}(D^n) \rightarrow e$  is a homeomorphism.
2.  $\varphi : D \rightarrow \overline{e}$  is a quotient map.

For simplicity, we denote an  $n$ -cell structure  $(D, \varphi)$  on  $e$  by  $e$  when there is no risk of confusion.  $D$  is called the *domain* of  $e$ . The number  $n$  is called the *dimension* of  $e$ .

<sup>10</sup>We say a covering  $X = \bigcup_{\lambda \in \Lambda} X_\lambda$  is *locally finite* if, for any  $x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $U \cap X_\lambda \neq \emptyset$  for only a finite number of  $\lambda$ .

**Remark 2.28.** The map  $\varphi$  is called the *characteristic map* of  $e$  when  $X$  is a cell complex. We prefer to call it the *cell structure map* for  $e$ .

**Example 2.29.** The open  $n$ -disk  $\text{Int}(D^n)$  is a globular  $n$ -cell. The standard homeomorphism

$$\text{Int}(D^n) \xrightarrow{\cong} \mathbb{R}^n$$

defines an  $n$ -cell structure on  $\mathbb{R}^n$ . The domain is  $\text{Int}(D^n)$ .  $\square$

The quotient topology condition is necessary for one of our main theorems (Theorem 4.15) to hold. Note that the quotient topology condition is missing in the definition of cell structures in [BGRT], since we required the regularity when we used cellular stratifications there. The quotient topology condition also guarantees that the cell structure map has the following maximality property.

**Lemma 2.30.** *Let  $(D, \varphi)$  be an  $n$ -cell structure on a subspace  $e \subset X$ . Then the pair  $(D, \varphi)$  is maximal in the poset of pairs satisfying the first condition of  $n$ -cell structure on  $e$  under inclusions.*

*Proof.* Let  $\tilde{\varphi} : \tilde{D} \rightarrow \bar{e}$  be a continuous map satisfying the first condition in Definition 2.27 together with the conditions that  $\tilde{D} \supset D$  and  $\tilde{\varphi}|_D = \varphi$ . Suppose there exists a point  $x \in \tilde{D} - D$ . Consider the subset

$$A = \left\{ \left(1 - \frac{1}{n}\right)x \mid n \in \mathbb{N} \right\}$$

of  $D$ . Since  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)x = x \notin D$ ,  $A = \varphi^{-1}(\varphi(A))$  is closed in  $D$ . Thus  $\varphi(A)$  is closed in  $\bar{e}$  by the quotient map condition on  $\varphi$ . On the other hand, the closure of  $A$  in  $\tilde{D}$  is  $A \cup \{x\}$  and the continuity of  $\tilde{\varphi}$  implies that

$$\overline{\varphi(A)} = \overline{\tilde{\varphi}(A)} \supset \tilde{\varphi}(\text{the closure of } A \text{ in } \tilde{D}) = \tilde{\varphi}(A \cup \{x\}) = \varphi(A) \cup \{\tilde{\varphi}(x)\}.$$

This contradicts to the closedness of  $\varphi(A)$  in  $\bar{e}$ . Thus  $\tilde{D} = D$ .  $\square$

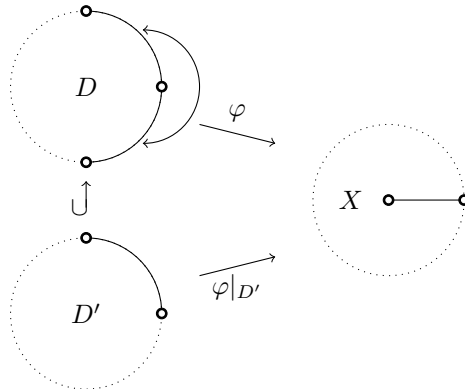
**Example 2.31.** Let  $X = \text{Int}(D^2) - \{(0, 0)\}$ ,  $e = X - \{(x, 0) \mid x > 0\}$  and  $D = \text{Int}(D^2) \cup S_+^1 - \{(1, 0)\}$ , where  $S_+^1 = \{(x, y) \in S^1 \mid x > 0\}$ . Then, by gluing the two open arcs in  $D$ , we obtain a map

$$\varphi : D \longrightarrow X$$

which is a homeomorphism onto  $e$  when restricted to  $\text{Int}(D^2)$ . We do not need two arcs for the first condition in Definition 2.27. Let  $D' = \text{Int}(D^2) \cup \{(x, y) \in S^1 \mid x > 0, y > 0\}$  and we still get a map

$$\varphi|_{D'} : D' \longrightarrow X$$

which is a homeomorphism onto  $e$  when restricted to  $\text{Int}(D^2)$ .



We imposed the quotient map condition in Definition 2.27 in order to avoid this kind of ambiguity. And, more importantly, we need two arcs to recover the homotopy type of  $X$  under the construction we will introduce in §4.  $\square$

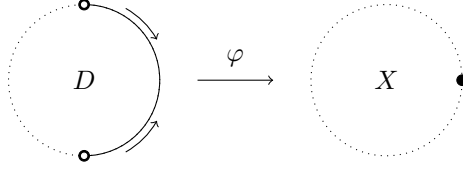
**Example 2.32.** Let  $X = \text{Int}(D^2) \cup \{(1, 0)\}$ . The identity map defines a 2-cell structure on  $X$ .

There is another choice. Let  $D = \text{Int}(D^2) \cup S_+^1$ . The deformation retraction of  $S_+^1$  onto  $(1, 0)$  can be extended to a continuous map

$$\varphi : D \longrightarrow X$$

whose restriction to  $\text{Int}(D^n)$  is a homeomorphism. For example,  $\varphi$  is given in polar coordinates by

$$\varphi(re^{i\theta}) = \begin{cases} re^{i(1-r)\theta}, & |\theta| \leq \frac{\pi}{2} \\ re^{i\{\theta - (\pi - \theta)r\}}, & \frac{\pi}{2} \leq \theta \leq \pi \\ re^{i\{\theta + (\pi + \theta)r\}}, & -\pi \leq \theta \leq -\frac{\pi}{2}. \end{cases}$$



Note that  $\varphi$  is not a quotient map. For example, the image of  $\{(x, y) \in D \mid x > 0\}$  under  $\varphi$  is open under the quotient topology, but it is not open under the relative topology on  $X$ . Thus this is not a 2-cell structure on  $X$ .  $\square$

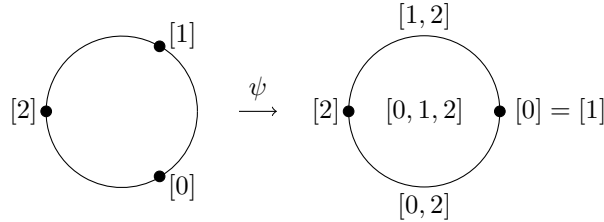
In other words, we need to put the quotient topology on  $X$  in order for the map  $\varphi$  in the above example to be a cell structure. Fortunately, we often use the quotient topology.

**Example 2.33.** For any simplicial set  $X$ , the geometric realization  $|X|$  is known to be a CW complex, whose cells are in one-to-one correspondence with nondegenerate simplices in  $X$ .

Consider the simplicial set  $X = s(\Delta^2)/s(\Delta^1)$ , where  $\Delta^1$  and  $\Delta^2$  are regarded as ordered simplicial complexes and  $s(-)$  is the functor in Example A.12 which transforms ordered simplicial complexes to simplicial sets. The geometric realization  $|X|$  is a cell complex consisting of two 0-cells  $[0] = [1]$ ,  $[2]$ , two 1-cells  $[0, 2]$ ,  $[1, 2]$ , and a 2-cell  $[0, 1, 2]$ . The cell structure map for the 2-cell is given by the composition

$$\psi : D^2 \cong \Delta^2 \times \{[0, 1, 2]\} \hookrightarrow \prod_{i=0}^{\infty} \Delta^n \times X_n \longrightarrow |X|.$$

Thus it is given by collapsing the arc  $[0, 1]$  in the picture below.



By definition,  $|X|$  is equipped with the quotient topology and  $\psi$  defines a 2-cell structure.  $\square$



Cells satisfying one (or both) of the following conditions appear frequently.

**Definition 2.34.** Let  $X$  be a topological space and  $e \subset X$  a subspace. An  $n$ -cell structure  $(D, \varphi)$  on  $e$ , or simply an  $n$ -cell  $e$ , is said to be

- *closed* if  $D = D^n$ ,
- *regular* if  $\varphi : D \rightarrow \bar{e}$  is a homeomorphism.

**Example 2.35.** Given a cell complex  $X$  and its  $n$ -cell  $e$ , the cell structure map

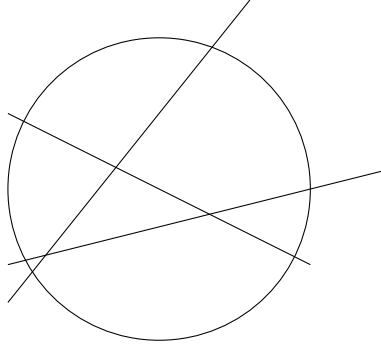
$$\varphi : D^n \longrightarrow X$$

may not be an  $n$ -cell structure, since it might not be a quotient map onto  $\bar{e}$ . When  $X$  is a CW complex, it is well known that  $\varphi : D^n \rightarrow$  is a quotient map and defines a closed  $n$ -cell structure on  $e$ . See Proposition A.2 in Hatcher's book [Hat02], for example.

When  $X$  is a regular CW complex, it is a regular  $n$ -cell structure on  $e$ .  $\square$

**Example 2.36.** Let  $\mathcal{A}$  be an arrangement of a finite hyperplanes in  $\mathbb{R}^n$ . Bounded strata in the stratification  $\pi_{\mathcal{A}}$  are convex polytopes and they are closed and regular cells.

We may also define non-closed but regular cell structures on unbounded strata as follows. Suppose  $\mathcal{A}$  is essential, namely the normal vectors to the hyperplanes span  $\mathbb{R}^n$ . Then we may choose a closed ball  $B$  with center at the origin in  $\mathbb{R}^n$  which contains all bounded strata.



Hyperplanes also cut the boundary sphere and define a stratification  $\pi_{\mathcal{A},B}$  on  $B$  whose strata are all closed cells. The inclusion  $\text{Int}(B) \hookrightarrow \mathbb{R}^n$  is a morphism of stratified spaces under which face posets can be identified

$$\begin{array}{ccc} \text{Int}(B) & \hookrightarrow & \mathbb{R}^n \\ \pi_{\mathcal{A},B} \downarrow & & \downarrow \pi_{\mathcal{A}} \\ P(\pi_{\mathcal{A},B}|_{\text{Int}(B)}) & \equiv & P(\pi_{\mathcal{A}}). \end{array}$$

For each unbounded stratum  $e$  in  $\pi_{\mathcal{A}}$ , the intersection  $e \cap \text{Int}(B)$  is a cell in  $B$ . Let  $\varphi : D^k \rightarrow \overline{e \cap B}$  be a cell structure map for  $e \cap \text{Int}(B)$  and define  $D = \varphi^{-1}(\bar{e} \cap \text{Int}(B))$ . We can compress the outside of  $B$  into  $e \cap \text{Int}(B)$  via a homeomorphism  $\psi : \bar{e} \rightarrow \bar{e} \cap \text{Int}(B)$ . The composition

$$D \xrightarrow{\varphi|_D} \bar{e} \cap \text{Int}(B) \xrightarrow{\psi^{-1}} \bar{e}$$

defines a regular cell structure on  $e$ . By an analogous argument, we can define cell structures on strata in the stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  defined in Example 2.10.  $\square$

Non-closed cells might have a bad topology.

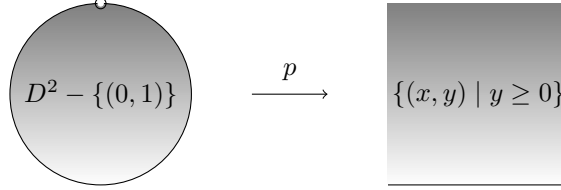
**Example 2.37.** Let

$$p : D^2 - \{(0, 1)\} \longrightarrow \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

be the homeomorphism given by extending the stereographic projection  $S^1 - \{(0, 1)\} \rightarrow \mathbb{R}$ . Let

$$X = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x, 0) \mid x \in \mathbb{Q}\}$$

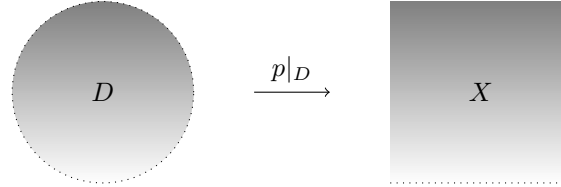
and define  $D = p^{-1}(X)$ .



Then the restriction

$$p|_D : D \longrightarrow X$$

defines a 2-cell structure on  $X$ .



$X$  and  $D$  are not locally compact. This example suggests that taking a product of cell structures might not be easy because of our requirement for a cell structure map to be a quotient map.

For example, let  $Y$  be the quotient space of  $X$  obtained by collapsing  $\{(n, 0) \mid n \in \mathbb{N}\}$  to a point. The composition of  $p|_D$  with the canonical projection

$$\varphi_Y : D \xrightarrow{p|_D} X \longrightarrow Y$$

defines a 2-cell structure on  $Y$ . But the product with  $p|_D$

$$p|_D \times \varphi_Y : D \times D \longrightarrow X \times Y$$

is not a quotient map, since the product  $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q} \times (\mathbb{Q}/\mathbb{N})$  of the identity map and the quotient map is not a quotient map. See Example 7 in p.143 of Munkres' book [Mun00].  $\square$

The same difficulty arises when we restrict cell structures to subspaces, since a restriction of a quotient map is not necessarily a quotient map.

**Definition 2.38.** Let  $X$  be a topological space,  $A$  a subspace, and  $\varphi : D \rightarrow \bar{e} \subset X$  a globular  $n$ -cell with  $e \subset A$ . When the restriction

$$\varphi|_{D_A} : D_A = \varphi^{-1}(\bar{e} \cap A) \longrightarrow \bar{e} \cap A$$

is a quotient map,  $\varphi|_{D_A}$  is called the *induced cell structure on  $e$  in  $A$* .

**Remark 2.39.** We discuss problems of taking products and restrictions in the sequel to this paper [Tama].

### 2.3 Cellular Stratifications

So far we have defined the notions of stratified spaces and cell structures. Now we are ready to define cellular stratified spaces by combining these two structures.

**Definition 2.40.** Let  $X$  be a Hausdorff space. A *cellular stratification* on  $X$  is a pair  $(\pi, \Phi)$  of a stratification  $\pi : X \rightarrow P(X)$  on  $X$  and a collection of cell structures  $\Phi = \{\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}\}_{\lambda \in P(X)}$  satisfying the condition that, for each  $n$ -cell  $e_\lambda$ ,  $\partial e_\lambda$  is covered by cells of dimension less than or equal to  $n - 1$ .

A *cellular stratified space* is a triple  $(X, \pi, \Phi)$  where  $(\pi, \Phi)$  is a cellular stratification on  $X$ . As usual, we abbreviate it by  $(X, \pi)$  or  $X$ , if there is no danger of confusion.

**Remark 2.41.** The term “cellular stratified space” has been already used in the study of singularities. See, for example, Schürmann’s book [Sch03]. We found, however, that his definition is too restrictive for our purposes.

**Example 2.42.** Consider the “topologist’s sine curve”

$$S = \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\}.$$

Its closure in  $\mathbb{R}^2$  is given by

$$\overline{S} = S \cup \{(0, t) \mid -1 \leq t \leq 1\}.$$

Since  $S$  is homeomorphic to the half interval  $(0, 1]$  via the function  $\sin \frac{1}{x}$ , the decomposition

$$\overline{S} = \{(1, \sin 1)\} \cup \{(0, 1)\} \cup \{(0, -1)\} \cup \{(x, \sin \frac{1}{x}) \mid 0 < x < 1\} \cup \{(0, t) \mid -1 < t < 1\}$$

is a stratification of  $\overline{S}$  consisting of five strata. Although the stratum  $\{(x, \sin \frac{1}{x}) \mid 0 < x < 1\}$  is homeomorphic to  $\text{Int}(D^1)$ , there is no 1-cell structure on this stratum, since there is no way to extend a homeomorphism  $(0, 1] \cong S$  to a continuous map  $[0, 1] \rightarrow \overline{S}$ .

Furthermore this stratification does not satisfy the dimension condition in Definition 2.40, since  $\partial S = \{(0, t) \mid -1 \leq t \leq 1\}$ .  $\square$

**Remark 2.43.** The above example is borrowed from Pflaum’s book [Pfl01]. He describes an even more pathological example. See 1.1.12 on page 18 of his book.

The following example says that cellular stratified spaces might have bad topology unless the CW conditions are imposed.

**Example 2.44.** Consider the 2-cell structure on  $X = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x, 0) \mid x \in \mathbb{Q}\}$  defined in Example 2.37. Let  $e^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  and  $e_x^0 = \{(x, 0)\}$  for  $x \in \mathbb{Q}$ . We have a decomposition

$$X = \bigcup_{x \in \mathbb{Q}} e_x^0 \cup e^2$$

and, for each  $x \in \mathbb{Q}$ , the identification

$$\psi_x : D^0 \cong \{(x, 0)\} \hookrightarrow X$$

defines a 0-cell structure on  $\{(x, 0)\}$ . Each  $e_x^0$  and  $e^2$  are locally closed and we obtain a cellular stratification.

This cellular stratified space does not satisfy the closure-finiteness condition, hence is not CW. It does not satisfy the weak-topology condition, either.  $\square$

**Definition 2.45.** We say a cellular stratification is *CW* if its underlying stratification is CW<sup>11</sup>.

The definition of cellular stratified spaces in this paper differs from the one in [BGRT] by the CW conditions.

**Lemma 2.46.** A CW cellular stratification  $(\pi, \Phi)$  on a space  $X$  defines and is defined by the following structure:

- a filtration  $X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset \cdots$  on  $X$ ,
- an  $n$ -cell structure on each connected component of  $X_n - X_{n-1}$

satisfying the following conditions:

1.  $X = \bigcup_{n=0}^{\infty} X_n$ .
2. For each stratum  $e$  in  $X_n$ ,  $\partial e$  is covered by a finite number of strata in  $X_{n-1}$ .
3.  $X$  has the weak topology determined by the covering  $\{\bar{e} \mid e \in P(X)\}$ .

*Proof.* Suppose  $(\pi, \Phi)$  is a CW cellular stratification. Define

$$X_n = \bigcup_{e \in P(X), \dim e \leq n} e;$$

then we obtain a filtration on  $X$  with  $X = \bigcup_{n=0}^{\infty} X_n$ . By definition, the boundary  $\partial e$  of each  $n$ -cell  $e$  is covered by a finite number of cells of dimension less than or equal to  $n-1$ . Also by definition,  $X$  has the weak topology determined by the covering  $\{\bar{e} \mid e \in P(X)\}$ .

It remains to show that the  $n$ -cells are the connected components of the difference  $X_n - X_{n-1}$ , i.e. each  $n$ -cell  $e$  is open and closed in  $X_n - X_{n-1}$ . By the local closedness,  $e$  is open in  $\bar{e}$ . Since  $\partial e \subset X_{n-1}$ ,  $e$  is open in  $X_n - X_{n-1}$ . For any  $n$ -cell  $e'$ , the intersection  $e \cap \bar{e}'$  in  $X_n - X_{n-1}$  is  $e \cap e'$  and is  $\emptyset$  or  $e = e'$ . And thus it is closed in  $\bar{e}'$ .

It is left to the reader to check the converse.  $\square$

**Example 2.47.** The stratification  $\pi_{\mathcal{A}}$  of  $\mathbb{R}^n$  defined by a hyperplane arrangement  $\mathcal{A}$  in Example 2.10 together with the cell structure defined in Example 2.36 is a CW cellular stratification.

The Björner-Ziegler stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  is also a CW cellular stratification.  $\square$

In Definition 2.15, we defined morphisms of stratified spaces as “stratification-preserving maps”. Note that, in our definition of cellular stratified spaces, we include cell structure maps as defining data. When we consider maps between cellular stratified spaces, we require them to be compatible with cell structure maps.

**Definition 2.48.** Let  $(X, \pi_X, \Phi_X)$  and  $(Y, \pi_Y, \Phi_Y)$  be cellular stratified spaces. A *morphism of cellular stratified spaces* from  $(X, \pi_X, \Phi_X)$  to  $(Y, \pi_Y, \Phi_Y)$  consists of

- a morphism  $f : (X, \pi_X) \rightarrow (Y, \pi_Y)$  of stratified spaces, and
- a family of maps

$$f_\lambda : D_\lambda \longrightarrow D_{f(\lambda)}$$

---

<sup>11</sup>Definition 2.19

indexed by cells  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  in  $X$  making the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi_\lambda \uparrow & & \uparrow \psi_{\underline{f}(\lambda)} \\ D_\lambda & \xrightarrow{f_\lambda} & D_{\underline{f}(\lambda)} \end{array}$$

commutative, where  $\psi_{\underline{f}(\lambda)} : D_{\underline{f}(\lambda)} \rightarrow \overline{e_{\underline{f}(\lambda)}}$  is the cell structure map for  $e_{\underline{f}(\lambda)}$ .

The category of cellular stratified spaces is denoted by **CSSpaces**.

**Remark 2.49.** When  $\mathbf{f} : (X, \pi_X, \Phi_X) \rightarrow (Y, \pi_Y, \Phi_Y)$  is a morphism of cellular stratified spaces, the compatibility of  $f_\lambda$  with cell structure maps implies  $f_\lambda(\text{Int}(D_\lambda)) \subset \text{Int}(D_{\underline{f}(\lambda)})$ .

**Remark 2.50.** In algebraic topology, the requirement for maps between cell complexes is much weaker. A map  $f : X \rightarrow Y$  between cell complexes is said to be *cellular*, if  $f(X_n) \subset Y_n$  for each  $n$ . The author thinks, however, this terminology is misleading.

**Definition 2.51.** A morphism  $(\mathbf{f}, \{f_\lambda\}) : (X, \pi_X, \Phi_X) \rightarrow (Y, \pi_Y, \Phi_Y)$  of cellular stratified spaces is said to be *strict* if  $\mathbf{f} : (X, \pi_X) \rightarrow (Y, \pi_Y)$  is a strict morphism of stratified spaces and  $f_\lambda(0) = 0$  for each  $\lambda \in P(X, \pi_X)$ .

**Definition 2.52.** Let  $(X, \pi)$  be a cellular stratified space. A stratified subspace  $(A, \pi|_A)$  of  $X$  is said to be a *cellular stratified subspace*, provided cell structures on cells in  $A$  are given as indicated in Definition 2.38. When the inclusion map  $A \hookrightarrow X$  is a strict morphism,  $A$  is said to be a *strict cellular stratified subspace*.

Once we have morphisms between cellular stratified spaces, we have a notion of equivariant cellular stratification.

**Definition 2.53.** Let  $G$  be a group. A cellular stratified space  $X$  equipped with a monoid morphism  $G \rightarrow \mathbf{CSSpaces}(X, X)$ <sup>12</sup> is called a *G-cellular stratified space*.

## 2.4 Stellar Stratified Spaces

In Definition 2.27, we required the domain  $D$  of an  $n$ -cell to contain  $\text{Int}(D^n)$ . As we will see in the proof of Theorem 4.15, this condition is requiring too much. Furthermore, the definition of the globular dimension of a cell is not appropriate when the cell is not closed. In this section, we introduce stellar cells and study stratified spaces whose strata are stellar cells. Stellar structures also play an essential role in extending the duality in classical simplicial topology to cellular stratified spaces, which is the subject of a separate paper [Tama].

Let us first define “star-shaped cells”.

**Definition 2.54.** A subset  $S$  of  $D^N$  is said to be an *aster* if  $\{0\} * \{x\} \subset S$  for any  $x \in S$ , where  $*$  is the join operation defined by connecting points by line segments<sup>13</sup> and 0 is the origin of  $D^N$ . The subset  $S \cap \partial D^N$  is called the *boundary* of  $S$  and is denoted by  $\partial S$ . The complement  $S - \partial S$  of the boundary is called the *interior* of  $S$  and is denoted by  $\text{Int}(S)$ .

We require the existence of a cellular stratification on the boundary in order to define the dimension.

<sup>12</sup>Or a functor  $G \rightarrow \mathbf{CSSpaces}$ .

<sup>13</sup>Definition A.25.

**Definition 2.55.** A *stellar cell* is an aster  $S$  in  $D^N$  for some  $N$  such that there exists a structure of regular cell complex on  $\overline{S}$  which contains  $\partial S$  as a strict cellular stratified subspace<sup>14</sup>.

When the (globular) dimension of  $\partial S$  is  $n - 1$ , we define the *stellar dimension* of  $S$  to be  $n$  and call  $S$  a *stellar  $n$ -cell*.

**Definition 2.56.** An  *$n$ -stellar structure* on a subset  $e$  of a topological space  $X$  is a pair  $(S, \varphi)$  of a stellar  $n$ -cell  $S$  and a continuous map

$$\varphi : S \longrightarrow X$$

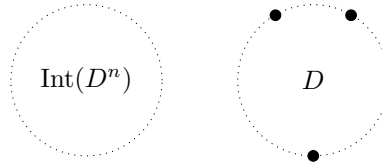
satisfying the following conditions:

1.  $\varphi(S) = \overline{e}$  and  $\varphi : S \rightarrow \overline{e}$  is a quotient map.
2. The restriction of  $\varphi$  to  $\text{Int}(S)$  is a homeomorphism onto  $e$ .

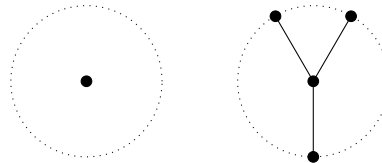
We say  $e$  is *thin* if  $S = \{0\} * \partial S$ . When  $e$  is thin and  $S$  is compact,  $e$  is said to be *closed*.

An  $n$ -cell in the sense of §2.2 is stellar if its boundary is a cellular stratified subspace of a regular cell decomposition of  $\partial D^n$ . However, the dimension as a stellar cell might be smaller than  $n$ .

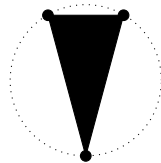
**Example 2.57.** Consider the globular  $n$ -cell  $\text{Int}(D^n)$  in Example 2.29. It is a stellar cell with empty boundary. Thus its stellar dimension is 0. By adding three points to the boundary, for example, we obtain a globular  $n$ -cell  $D$  whose stellar dimension is 1.



These two stellar cells are not thin, hence not closed. The first example contains  $\{0\}$  as a closed stellar cell. The second example contains a graph of the shape of  $Y$  as a closed stellar cell.



A 2-simplex inscribed in the unit 2-disk is a stellar 1-cell. This is a compact aster but is not thin, hence is not closed.



□

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<sup>14</sup>Definition 2.52

The proof of Lemma 2.30 can be used without a change to prove the following analogous fact.

**Lemma 2.58.** *Let  $(S, \varphi)$  be an  $n$ -stellar structure on a subspace  $e \subset X$ . Then the pair  $(S, \varphi)$  is maximal among the pairs satisfying the first condition of  $n$ -stellar structure on  $e$ .*

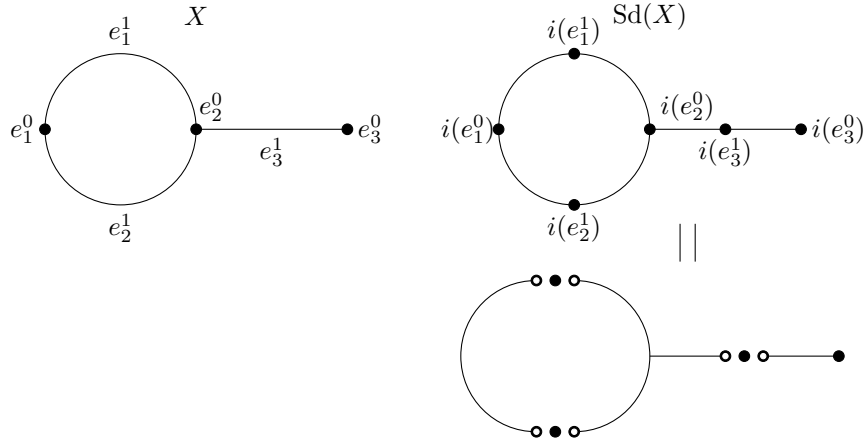
By replacing cell structures by stellar structures in the definition of cellular stratifications, we obtain the notion of stellar stratifications.

**Definition 2.59.** A *stellar stratification* on a topological space  $X$  consists of a stratification  $(X, \pi)$ , and a stellar structure on each  $e_\lambda = \pi^{-1}(\lambda)$  for  $\lambda \in P(X)$  satisfying the condition that for each stellar  $n$ -cell  $e_\lambda$ ,  $\partial e_\lambda$  is covered by stellar cells of stellar dimension less than or equal to  $n - 1$ .

A space equipped with a stellar stratification is called a *stellar stratified space*. When all stellar cells in  $X$  are closed,  $X$  is called a *stellar complex*.

The following is a typical example.

**Example 2.60.** Consider a finite graph  $X$  regarded as a 1-dimensional cell complex. The barycentric subdivision  $\text{Sd}(X)$  of  $X$  can be expressed as a union of open stars of vertices of  $X$  and the barycenters  $i(e_1^1)$ ,  $i(e_2^1)$ ,  $i(e_3^1)$  of 1-cells in  $X$ .



And we have a structure of stellar stratified space on  $\text{Sd}(X)$ . □

**Remark 2.61.** We investigate more precise relations between stellar stratifications and the barycentric subdivision of cellular stratified spaces in [Tama].

The definition of morphisms of stellar stratified spaces should be obvious.

**Definition 2.62.** The category of stellar stratified spaces is denoted by **SSSpaces**.

### 3 “Niceness Conditions” on Cellular and Stellar Stratified Spaces

In order to be practical, we need to impose appropriate niceness conditions on cellular and stellar stratified spaces.

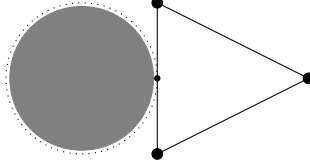
### 3.1 Regularity and Normality

Regularity and normality are frequently used conditions on CW complexes. We have already defined normality for stratified spaces in Definition 2.7 and regularity for cells in Definition 2.34.

**Definition 3.1.** Let  $X$  be a cellular or stellar stratified space. We say  $X$  is *normal*, if it is normal as a stratified space. We say  $X$  is *regular* if all cells in  $X$  are regular.

**Example 3.2.** The cellular stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  in Example 2.10 defined by a real arrangement  $\mathcal{A}$  in  $\mathbb{R}^n$  is regular and normal.  $\square$

**Example 3.3.** Consider the following cellular stratified space



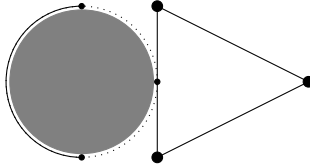
obtained by gluing  $\text{Int}(D^2)$  to the boundary of a 2-simplex at the middle point of an edge. The domain of the cell structure map of the 2-cell is  $\text{Int}(D^2) \cup \{(1, 0)\}$ , whose boundary is mapped into the 1-skeleton. Thus this is a regular cellular stratified space. However, this is not normal.

If we regard the globular 2-cell  $\text{Int}(D^2) \cup \{(1, 0)\}$  as a stellar cell, its stellar dimension is 1 and this stellar structure does not satisfy the dimensional requirement. Thus this is not a stellar stratified space.  $\square$

In the case of CW complexes, regularity always implies normality. See Theorem 2.1 in Chapter III of the book [LW69] by Lundell and Weingram. The above example suggests that the failure of this fact for cellular stratified spaces is partly due to the “wrong definition” of dimensions of globular cells. The right notion of dimension is the stellar one.

Even for stellar stratified spaces, however, regularity does not necessarily imply normality.

**Example 3.4.** By adding an arc to the previous example, we obtain a stratified space as follows.



The globular 2-cell  $\text{Int}(D^2) \cup \{(1, 0)\} \cup \{(x, y) \in S^1 \mid x \leq 0\}$  is a stellar 2-cell. And this is a regular stellar stratified space. But this is not normal.  $\square$

**Remark 3.5.** It seems that, if  $X$  is a stellar stratified space in which the boundary of each stellar  $n$ -cell is “pure of dimension  $n - 1$ ” and if  $X$  is regular, then  $X$  is normal.

We make use of lifts of cell structure maps in order to understand relations among cells.

**Definition 3.6.** Let  $X$  be a cellular stratified space. For each pair  $\mu < \lambda$ , define  $F(X)(e_\mu, e_\lambda)$  to be the set of all maps  $b : D_\mu \rightarrow D_\lambda$  making the diagram

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \overline{e_\lambda} \\ \uparrow b & & \uparrow \text{J} \\ D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_\mu} \end{array}$$



commutative, where  $\varphi_\lambda$  and  $\varphi_\mu$  are the cell structure maps of  $e_\lambda$  and  $e_\mu$ , respectively.

We say that  $X$  is *strongly normal* if  $F(X)(e_\mu, e_\lambda) \neq \emptyset$  for all pairs  $\mu < \lambda$ .

The following definition was introduced in [BGR] in order to describe a condition under which the order complex of the face poset of a regular cellular stratified space is homotopy equivalent to the original space.

**Definition 3.7.** Let  $X$  be a normal cellular stratified space.  $X$  is called *totally normal* if, for each  $n$ -cell  $e_\lambda$ ,

1. there exists a structure of regular cell complex on  $S^{n-1}$  containing  $\partial D_\lambda$  as a strict cellular stratified subspace<sup>15</sup> of  $S^{n-1}$  and
2. for any cell  $e$  in  $\partial D_\lambda$ , there exists a cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the cell structure map of  $e_\mu$  factors through  $D_\lambda$  via the cell structure map of  $e$ :

$$\begin{array}{ccccc}
 \overline{e} & \xrightarrow{\quad} & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & X \\
 \uparrow & & & & & \nearrow \varphi_\mu & \\
 D & \xlongequal{\quad} & D_\mu & & & & 
 \end{array}$$

Total normality for stellar stratified spaces is defined analogously. But we need to use cellular subdivisions.

**Definition 3.8.** Let  $X$  be a normal stellar stratified space. We say  $X$  is *totally normal* if, for each stellar  $n$ -cell  $e_\lambda$  with domain  $D_\lambda \subset D^N$ ,

1. there exist a structure of stellar stratified space on  $\partial D_\lambda$  such that each stellar cell in  $\partial D_\lambda$  is a strict stratified subspace of the regular cell complex  $\overline{\partial D_\lambda}$ <sup>16</sup>, and
2. for any stellar cell  $e$  in  $\partial D_\lambda$ , there exists a stellar cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the cell structure map of  $e_\mu$  factors through  $D_\lambda$  via the cell structure map of  $e$ .

Because of the existence of a regular cell decomposition on the boundary of the domain of each cell, we have the following fact.

**Lemma 3.9.** *Any totally normal cellular stratified space has a structure of a totally normal stellar stratified space.*

The cell structure maps of totally normal stellar stratified spaces preserve cells.

**Lemma 3.10.** *For a cell  $e_\lambda$  in a totally normal stellar stratified space  $X$ , let  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda} \subset X$  be the cell structure map. Then there exists a structure of stellar stratified space on  $D_\lambda$  under which  $\varphi_\lambda$  is a strict morphism of stellar stratified spaces.*

*Proof.* Let  $e_\lambda$  be a cell in a totally normal stellar stratified space  $X$  and  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  the cell structure map. Let

$$\partial D_\lambda = \bigcup_{\nu} e'_\nu$$

<sup>15</sup>Definition 2.52.

<sup>16</sup>Recall that  $\overline{\partial D_\lambda}$  is a finite regular cell complex embedded in  $S^{N-1}$  (Definition 2.56).

be the stellar stratification in the definition of total normality. We have

$$\partial e_\lambda = \varphi_\lambda(\partial D_\lambda) = \bigcup_{\nu} \varphi_\lambda(e'_\nu).$$

By the definition of total normality, for each  $\nu$ , there exists a stellar cell  $e_{\mu'}$  in  $\partial e_\lambda$  whose stellar structure map makes the diagram

$$\begin{array}{ccccc} \overline{e'_\nu} & \hookrightarrow & \partial D_\lambda & \xrightarrow{\varphi_\lambda} & \partial e_\lambda \\ \uparrow \psi_\nu & & & & \uparrow \\ D_\nu & \xlongequal{\quad} & D_{\mu'} & \xrightarrow{\varphi_{\mu'}} & \overline{e_{\mu'}}, \end{array}$$

commutative, where  $\psi_\nu$  is the stellar structure map for  $e'_\nu$ . This implies that each  $\varphi_\lambda(e'_\nu)$  is a stellar cell in  $X$  and thus  $\varphi_\lambda$  is a strict morphism of stellar stratified spaces.  $\square$

**Corollary 3.11.** *Let  $(\pi, \Phi)$  be a stellar stratification on  $X$  satisfying the first condition of total normality.*

*Then it is totally normal if and only if*

$$\partial D_\lambda = \bigcup_{e_\mu < e_\lambda} \bigcup_{b \in F(X)(e_\mu, e_\lambda)} b(\text{Int}(D_\mu)). \quad (1)$$

*as stellar stratified spaces.*

*Proof.* Suppose  $(\pi, \Phi)$  is totally normal. For a pair of cells  $e_\mu < e_\lambda$ , by Lemma 3.10, there exists a stellar cell  $e$  in  $D_\lambda$  such that  $\varphi_\lambda(e) = e_\mu$ . By the assumption of total normality, the cell structure map  $\psi : D \rightarrow \overline{e}$  of  $e$  makes the following diagram commutative:

$$\begin{array}{ccccc} \overline{e} & \hookrightarrow & \partial D_\lambda & \xrightarrow{\varphi_\lambda} & \partial e_\lambda \\ \uparrow \psi & & & & \uparrow \\ D & \xlongequal{\quad} & D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_\mu} \end{array}$$

The collection of all such cell structure maps  $\psi$  is  $F(X)(e_\mu, e_\lambda)$  and we have

$$\partial D_\lambda = \bigcup_{e_\mu < e_\lambda} \bigcup_{b \in F(X)(e_\mu, e_\lambda)} b(\text{Int}(D_\mu)).$$

Conversely, the assumption (1) implies that, for any stellar cell  $e$  in  $\partial D_\lambda$ , there is a corresponding cell  $e_\mu$  in  $\partial e_\lambda$  whose cell structure map makes the required diagram commutative.  $\square$

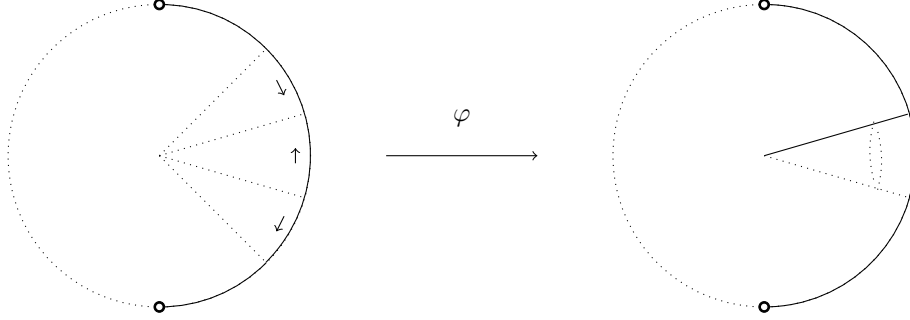
The lifts  $b : D_\mu \rightarrow D_\lambda$  of cell structure maps appeared in the above Corollary play an essential role when we analyze totally normal stellar stratified spaces. It is easy to see that each  $b$  is an embedding.

**Lemma 3.12.** *Let  $X$  be a totally normal stellar stratified space. Then each  $b \in F(X)(e_\mu, e_\lambda)$  is an embedding of stellar stratified spaces for each pair  $e_\mu < e_\lambda$ .*

*Proof.* This follows from the assumption that the cellular stratification on  $D_\lambda$  is regular.  $\square$

Let us take a look at a couple of examples. The first example is borrowed from Kirillov's paper [Kir].

**Example 3.13.** Consider  $D = \text{Int}(D^2) \cup S_+^1 = e^1 \cup e^2$ . Divide the 1-cell into five arcs. Define  $X$  by folding the middle three arcs in  $e^1$  according to the directions indicated by the arrows in the figure below.



Note that  $\varphi(e^1)$  is homeomorphic to  $\text{Int}(D^1)$ . Let

$$\psi : \text{Int}(D^1) \longrightarrow e^1 \subset X$$

be a homeomorphism. Identifications only occur on  $e^1$  and the quotient map

$$\varphi : D \longrightarrow X$$

is a homeomorphism onto its image when restricted to  $\text{Int}(D^2)$  and thus defines a cell structure map for the 2-cell.

These maps  $\psi$  and  $\varphi$  define a cellular stratification on  $X$ . However, there is no way to obtain a map  $h : \text{Int}(D^1) \rightarrow D$  making the diagram commutative

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & X \\ \uparrow h & \nearrow \psi & \\ \text{Int}(D^1) & & \end{array}$$

and thus this cellular stratification is not totally normal. Note, however, that we obtain a totally normal cellular stratification by an appropriate subdivision of  $e^1$  and  $\varphi(e^1)$ .  $\square$

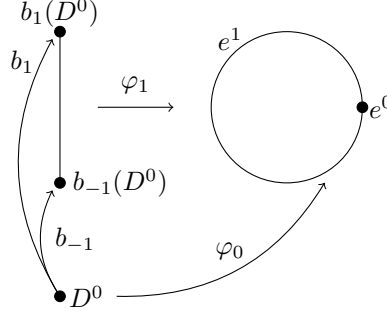
**Example 3.14.** Consider the minimal cell decomposition

$$S^1 = e^0 \cup e^1 = \{(1, 0)\} \cup (S^1 - \{(1, 0)\}).$$

The cell structure map for the 1-cell

$$\varphi_1 : D^1 = [-1, 1] \longrightarrow S^1$$

is given by  $\varphi_1(t) = (\cos(\pi(t + 1)), \sin(\pi(t + 1)))$ . There are two lifts  $b_{-1}$  and  $b_1$  of the cell structure map  $\varphi_0$  for the 0-cell.



Since  $\partial D^1 = \{-1, 1\} = b_{-1}(D^0) \cup b_1(D^0)$ , this is totally normal.  $\square$

**Example 3.15.** More generally, any 1-dimensional CW-complex is totally normal. In fact, 0-cells are always regular and there are only two types of 1-cells, i.e. regular cells and cells whose cell structure maps are given by collapsing the boundary of  $D^1$  to a point. By the above example, all 1-cells are totally normal. This fact allows us to apply results of this paper to configuration spaces of graphs [FT].  $\square$

**Example 3.16.** Consider the punctured torus

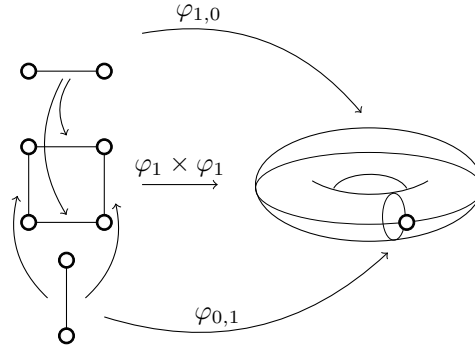
$$X = S^1 \times S^1 - e^0 \times e^0 = (e^1 \times e^0) \cup (e^0 \times e^1) \cup (e^1 \times e^1)$$

with the stratification induced from the product stratification of the minimal cell decomposition of  $S^1$ .

A cell structure map for the 2-cell can be obtained by removing four corners from the domain of the product

$$\varphi_1 \times \varphi_1 : D^1 \times D^1 - \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \longrightarrow X$$

of  $\varphi_1$  in Example 3.14. Let us denote the cell structure maps for  $e^1 \times e^0$  and  $e^0 \times e^1$  by  $\varphi_{1,0}$  and  $\varphi_{0,1}$ , respectively.



The cell structure map of each 1-cell has two lifts and the images cover the boundary of the domain of the 2-cell. Thus this is totally normal.  $\square$

**Example 3.17.** Let  $X$  be a  $\Delta$ -set<sup>17</sup>, i.e. a functor  $X : \Delta_{\text{inj}}^{\text{op}} \rightarrow \mathbf{Sets}$ , where  $\Delta_{\text{inj}}$  is the subcategory of the category  $\Delta$  of (isomorphism classes of) finite totally ordered sets consisting of injective

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<sup>17</sup>Definition A.14.

maps. Define

$$\pi_X : \|X\| \longrightarrow \coprod_{n=0}^{\infty} X_n$$

by  $\pi_X(x) = \sigma$  if  $x$  is represented by  $(\mathbf{t}, \sigma) \in \Delta^n \times X_n$  with  $t_i \neq 0$  for all  $i \in \{0, \dots, n\}$ . Note that  $P(\|X\|) = \coprod_{n=0}^{\infty} X_n$  can be made into a poset by defining  $\tau \leq \sigma$  if and only if there exists a morphism  $u : [m] \rightarrow [n]$  in  $\Delta_{\text{inj}}$  with  $X(u)(\sigma) = \tau$ .

Then the map  $\pi_X$  is a cellular stratification and we have

$$\|X\| = \bigcup_{n=0}^{\infty} \bigcup_{\sigma \in X_n} \text{Int}(\Delta^n) \times \{\sigma\}.$$

Let us denote the  $n$ -cell  $\text{Int}(\Delta^n) \times \{\sigma\}$  corresponding to  $\sigma \in X_n$  by  $e_\sigma$ . The cell structure map  $\varphi_\sigma$  for  $e_\sigma$  is defined by the composition

$$\varphi_\sigma : D^n \cong \Delta^n \times \{\sigma\} \hookrightarrow \coprod_{n=0}^{\infty} \Delta^n \times X_n \longrightarrow \|X\|.$$

Suppose  $\tau < \sigma$  for  $\tau \in X_m$  and  $\sigma \in X_n$ . There exists a morphism  $u : [m] \rightarrow [n]$  with  $X(u)(\sigma) = \tau$ . With this morphism, we have the commutative diagram

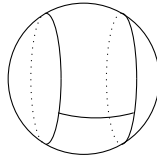
$$\begin{array}{ccc} \Delta^n \times \{\sigma\} & \xrightarrow{\varphi_\sigma} & \overline{e_\sigma} \\ \uparrow u_*^{\tau, \sigma} & & \uparrow \\ \Delta^m \times \{\tau\} & \xrightarrow{\varphi_\tau} & \overline{e_\tau}, \end{array}$$

where  $u_*^{\tau, \sigma} : \Delta^m \times \{\tau\} \rightarrow \Delta^n \times \{\sigma\}$  is a copy of the affine map  $u_* : \Delta^m \rightarrow \Delta^n$  induced by the inclusion of vertices  $u : [m] \rightarrow [n]$ . Hence this cellular stratification on  $\|X\|$  is totally normal.

In particular, for any acyclic category  $C$  (with discrete topology), the classifying space  $BC$  is a totally normal CW complex by Lemma B.13.  $\square$

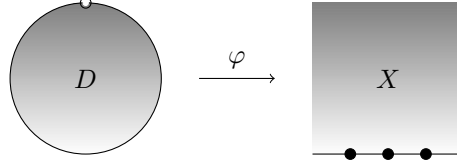
**Example 3.18.** A. Kirillov, Jr. introduced a structure called PLCW complex in [Kir]. A PLCW complex is defined by attaching PL-disks. The attaching map of an  $n$ -cell is required to be a strict morphism of cellular stratified spaces under a suitable PLCW decomposition of the boundary sphere.

Besides the PL requirement in Kirillov's definition, the only difference between totally normal CW complexes and PLCW complexes is that, in totally normal CW complexes, the boundary sphere of each cell structure map is required to have a regular cell decomposition. For example, the following cell decomposition of  $D^3$  is allowed as the domain of a 3-cell in a PLCW complex but is not allowed in a totally normal cellular stratified space.



$\square$

**Example 3.19.** Let  $X = \mathbb{R} \times \mathbb{R}_{\geq 0}$  with 0-cells  $e_n^0 = \{(n, 0)\}$  for  $n \in \mathbb{Z}$ , 1-cells  $e_n^1 = (n, n+1) \times \{0\}$  for  $n \in \mathbb{Z}$ , and a 2-cell  $e^2 = \mathbb{R} \times \mathbb{R}_{>0}$ . The cell structure map  $\varphi$  of the 2-cell is given by extending the stereographic projection  $S^1 - \{(0, 1)\} \rightarrow \mathbb{R}$ . The domain is  $D = D^2 - \{(0, 1)\}$ .



This is regular and normal. It even satisfies the second condition in Definition 3.7. But it is not totally normal, since  $D^2$  is compact.

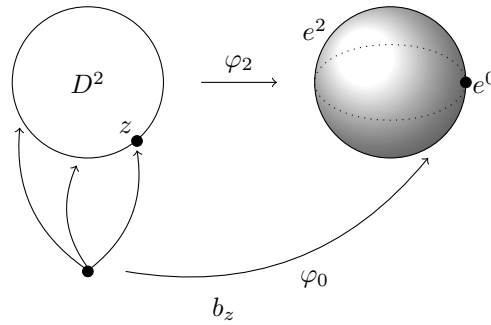
Note that this example does not satisfy the closure finiteness condition. Hence it is not CW.  $\square$

**Example 3.20.** Consider the minimal cell decomposition

$$S^2 = e^0 \cup e^2.$$

We may choose a regular cell decomposition of  $\partial D^2$  and make it a stellar stratification. For example, we use the minimal regular cell decomposition  $S^1 = e_+^0 \cup e_-^0 \cup e_+^1 \cup e_-^1$ .

In order to make  $S^2$  totally normal, however, this stratification is not fine enough. There are infinitely many lifts of the cell structure map of the 0-cell parametrized by points in  $S^1$ .



Although  $\partial D^2$  is covered by the images of the maps  $b_z$ 's

$$\partial D^2 = \bigcup_{z \in S^1} b_z(D^0),$$

this is not a cell decomposition of  $\partial D^2$ . Hence this is not totally normal.  $\square$

## 3.2 Cylindrical Structures

In Example 3.20, lifts of  $\varphi_0$  are parametrized by points in  $S^1 = \partial D^2$ . This example suggests that we need to topologize the set of all lifts. Inspired by the work of Cohen, Jones, and Segal on Morse theory [CJS] and this example, we introduce the following definition.

**Definition 3.21.** A *cylindrical structure* on a normal cellular stratified space  $(X, \pi)$  consists of

- a normal stratification on  $\partial D^n$  containing  $\partial D_\lambda$  as a stratified subspace for each  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e}_\lambda$ ,

- a stratified space  $P_{\mu,\lambda}$  and a morphism of stratified spaces

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$

for each pair of cells  $e_\mu \subset \partial e_\lambda$ , and

- a morphism of stratified spaces

$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}$$

for each sequence  $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$

satisfying the following conditions:

1. The restriction of  $b_{\mu,\lambda}$  to  $P_{\mu,\lambda} \times \text{Int}(D_\mu)$  is a homeomorphism onto its image.
2. The following three types of diagrams are commutative:

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & X \\ b_{\mu,\lambda} \uparrow & & \uparrow \varphi_\mu \\ P_{\mu,\lambda} \times D_\mu & \xrightarrow{\text{pr}_2} & D_\mu \end{array}$$
  

$$\begin{array}{ccc} P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \times D_{\lambda_0} & \xrightarrow{1 \times b_{\lambda_0,\lambda_1}} & P_{\lambda_1,\lambda_2} \times D_{\lambda_1} \\ c_{\lambda_0,\lambda_1,\lambda_2} \times 1 \downarrow & & \downarrow b_{\lambda_1,\lambda_2} \\ P_{\lambda_0,\lambda_2} \times D_{\lambda_0} & \xrightarrow{b_{\lambda_0,\lambda_2}} & D_{\lambda_2} \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} & \xrightarrow{c \times 1} & P_{\lambda_1,\lambda_3} \times P_{\lambda_0,\lambda_1} \\ 1 \times c \downarrow & & \downarrow c \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_0,\lambda_2} & \xrightarrow{c} & P_{\lambda_0,\lambda_3} \end{array}$$

3. We have

$$\partial D_\lambda = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(P_{\mu,\lambda} \times \text{Int}(D_\mu))$$

as a stratified space.

The space  $P_{\mu,\lambda}$  is called the *parameter space* for the inclusion  $e_\mu \subset \overline{e_\lambda}$ . When  $\mu = \lambda$ , we define  $P_{\lambda,\lambda}$  to be a single point. A cellular stratified space equipped with a cylindrical structure is called a *cylindrically normal cellular stratified space*.

When the map  $b_{\lambda,\mu}$  is an embedding for each pair  $e_\mu \subset \overline{e_\lambda}$ , the stratification is said to be *strictly cylindrical*.

**Remark 3.22.** The author first intended to call such a structure a “locally product-like” or “locally trivial” cellular stratification. But it turns out the term “locally trivial stratification” is already used in [Pff01] in a different sense.

Cylindrical structures on stellar stratified spaces are defined analogously. The only difference is that the domain  $D_\lambda$  of a stellar  $n$ -cell  $e_\lambda$  may not be a subspace of  $D^n$  but of a disk  $D^N$  of a larger dimension.

**Definition 3.23.** A *cylindrical structure* on a normal stellar stratified space  $(X, \pi)$  consists of

- a coarsening<sup>18</sup> of the stratification on  $\partial D_\lambda$  in the definition of stellar structure for each stellar  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ ,
- a stratified space  $P_{\mu,\lambda}$  and a morphism of stratified spaces

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$

for each pair of cells  $e_\mu \subset \partial e_\lambda$ , and

- a morphism of stratified spaces

$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}$$

for each sequence  $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$

satisfying the three conditions in Definition 3.21.

**Example 3.24.** A stellar stratified space  $X$  is totally normal if and only if it is strictly cylindrically normal and each parameter space  $P_{\mu,\lambda}$  is a finite set (with discrete topology).

Consider the cell decomposition of  $D^3$  in Example 3.18. It is easily seen to be cylindrically normal with finite parameter spaces. However, it is not strictly cylindrical, as we have seen in Example 3.18. In other words, a cylindrically normal cell complex with finite parameter spaces is a PLCW complex, if it satisfies the PL requirement in the definition of PLCW complexes.  $\square$

**Example 3.25.** Consider the minimal cell decomposition of  $S^2 = \mathbb{CP}^1$  in Example 3.20.

If we regard  $S^2$  as a cellular stratified space, the trivial stratification on  $\partial D^2$  and the canonical inclusion

$$b_{0,2} : S^1 \times D^0 \longrightarrow \partial D^2 \subset D^2$$

define a cylindrical structure on  $S^2$  with  $P_{0,2} = S^1$ , for we have a commutative diagram

$$\begin{array}{ccc} D^2 & \xrightarrow{\varphi_2} & \mathbb{CP}^1 \\ \uparrow b_{0,2} & & \uparrow \varphi_0 \\ S^1 \times D^0 & \xrightarrow{\text{pr}_2} & D^0. \end{array}$$

We may also consider  $S^2$  as a stellar stratified space, by choosing a regular cell decomposition of  $\partial D^2$ . For example, we may use the minimal regular cell decomposition of  $\partial D^2$ .

Then we have embeddings

$$\begin{aligned} b_{0,+} & : D^0 \times D^0 \longrightarrow \partial D^2 \\ b_{0,-} & : D^0 \times D^0 \longrightarrow \partial D^2 \\ b_{1,+} & : D^0 \times D^1 \longrightarrow \partial D^2 \\ b_{1,-} & : D^0 \times D^1 \longrightarrow \partial D^2 \end{aligned}$$

corresponding to cells  $e_+^0 \cup e_-^0 \cup e_+^1 \cup e_-^1$  in  $\partial D^2$ . And we obtain a cylindrical structure as a stellar stratified space.  $\square$

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<sup>18</sup>Definition 2.15



**Example 3.26.** Let us extend the cylindrically normal cell decomposition on  $S^2 = \mathbb{CP}^1$  to  $\mathbb{CP}^2$ . The minimal cell decomposition of  $\mathbb{CP}^2$  is given by

$$\mathbb{CP}^2 = S^2 \cup e^4 = e^0 \cup e^2 \cup e^4.$$

Consider the cell structure map of the 4-cell

$$\varphi_4 = \tilde{\eta} : D^4 \longrightarrow \mathbb{CP}^2$$

whose restriction to the boundary is the Hopf map

$$\eta : S^3 \longrightarrow S^2.$$

This is a fiber bundle with fiber  $S^1$  and thus the cell decomposition  $S^2 = e^0 \cup e^2$  induces a decomposition

$$S^3 \cong e^0 \times S^1 \cup e^2 \times S^1,$$

as we have seen in Example 2.17.

Let

$$\begin{aligned} \varphi_0 : D^0 &\rightarrow \overline{e^0} \subset S^2 \\ \varphi_2 : D^2 &\rightarrow \overline{e^2} \subset S^2 \end{aligned}$$

be the cell structure maps of  $e^0$  and  $e^2$ , respectively. We have a trivialization

$$t : \varphi_2^*(S^3) \xrightarrow{\cong} D^2 \times S^1.$$

Let  $b_{2,4} : S^1 \times D^2 \rightarrow S^3 = \partial D^4$  be the composition

$$S^1 \times D^2 \xrightarrow{t^{-1}} \varphi_2^*(S^3) \xrightarrow{\varphi_2} S^3$$

then we have the commutative diagram

$$\begin{array}{ccccc} S^3 & \xlongequal{\quad} & S^3 & \xrightarrow{\eta} & \mathbb{CP}^1 \\ & & \uparrow \varphi_2 & & \uparrow \varphi_2 \\ & & \varphi_2^*(S^3) & \longrightarrow & D^2 \\ & \nearrow t^{-1} & & & \parallel \\ S^1 \times D^2 & \xrightarrow{\text{pr}_2} & D^2. & & \end{array}$$

Let  $b_{0,4} : S^1 \times D^0 \rightarrow S^3$  be the inclusion of the fiber over  $e^0$ . Then we have

$$\partial D^4 = b_{0,4}(S^1 \times D^0) \cup b_{2,4}(S^1 \times D^2).$$

Let  $P_{0,2} = P_{2,4} = P_{0,4} = S^1$  and define

$$c_{0,2,4} : P_{2,4} \times P_{0,2} \rightarrow P_{0,4}$$

by the multiplication of  $S^1$ .

Let us check that the above data define a cylindrical structure on  $\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4$ . It remains to verify the commutativity of the diagram

$$\begin{array}{ccc} P_{2,4} \times P_{0,2} \times D^0 & \xrightarrow{1 \times b_{0,2}} & P_{2,4} \times D^2 \\ c \times 1 \downarrow & & \downarrow b_{2,4} \\ P_{0,4} \times D^0 & \xrightarrow{b_{0,4}} & D^4. \end{array}$$

In other words, we need to show the restriction of  $b_{2,4}$  to

$$b_{2,4}|_{S^1 \times S^1} : S^1 \times S^1 \longrightarrow S^3 = \partial D^4$$

is given by the multiplication of  $S^1$  followed by the inclusion of the fiber  $\eta^{-1}(e^0)$ . Recall that the Hopf map  $\eta$  is given by

$$\eta(z_1, z_2) = (2|z_1|^2 - 1, 2z_1\bar{z}_2),$$

where we regard

$$\begin{aligned} S^2 &= \{(x, z) \in \mathbb{R} \times \mathbb{C} \mid x^2 + |z|^2 = 1\} \\ S^3 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}, \end{aligned}$$

and, on  $U_+ = S^2 - \{(1, 0)\}$ , the local trivialization

$$\varphi_+ : \eta^{-1}(U_+) \longrightarrow U_+ \times S^1$$

is given by

$$\varphi_+(z_1, z_2) = \left( 2|z_1|^2 - 1, 2z_1\bar{z}_2, \frac{z_2}{|z_2|} \right).$$

The inverse of  $\varphi_+$  is given by

$$\varphi_+^{-1}(x, z, w) = \left( \frac{zw}{2\sqrt{\frac{1-x}{2}}}, w\sqrt{\frac{1-x}{2}} \right).$$

Define

$$\text{wrap} : D^2 \longrightarrow S^2$$

by

$$\text{wrap}(z) = \left( 2|z| - 1, 2\sqrt{|z|(1-|z|)} \frac{z}{|z|} \right);$$

then the restriction of  $\text{wrap}$  to  $\text{Int}(D^2)$  is a homeomorphism onto  $S^2 - \{(1, 0)\}$ . The map  $b_{2,4}$  is defined by the composition

$$S^1 \times \text{Int}(D^2) \xrightarrow{1 \times \text{wrap}} S^1 \times U_+ \cong U_+ \times S^1 \xrightarrow{\varphi_+^{-1}} \eta^{-1}(U_+) \hookrightarrow S^3,$$

which is given by

$$\begin{aligned}
b_{2,4}(w, z) &= \varphi_+^{-1} \left( 2|z| - 1, 2\sqrt{|z|(1-|z|)} \frac{z}{|z|}, w \right) \\
&= \left( \frac{2\sqrt{|z|(1-|z|)} \frac{z}{|z|} w}{2\sqrt{\frac{1-(2|z|-1)}{2}}}, w\sqrt{\frac{1-(2|z|-1)}{2}} \right) \\
&= \left( \frac{zw}{\sqrt{|z|}}, w\sqrt{1-|z|} \right).
\end{aligned}$$

From this calculation, we see  $b_{2,4}(w, z) \rightarrow (zw, 0)$  as  $|z| \rightarrow 1$ .

Thus this is a cylindrically normal cellular stratification.  $\square$

**Example 3.27.** There is an alternative way of describing the cylindrical structure in the above Example. Recall that complex projective spaces are typical examples of quasitoric manifolds. Define an action of  $T^n = (S^1)^n$  on  $\mathbb{CP}^n$  by

$$(t_1, \dots, t_n) \cdot [z_0, \dots, z_n] = [z_0, t_1 z_1, \dots, t_n z_n].$$

As we have seen in Example 2.12, this action induces a stratifications on  $\mathbb{CP}^n$  which descends to  $\mathbb{CP}^n/T^n$

$$\begin{aligned}
\pi_{\mathbb{CP}^n} &: \mathbb{CP}^n \longrightarrow I(T^n) \\
\pi_{\mathbb{CP}^n/T^n} &: \mathbb{CP}^n/T^n \longrightarrow I(T^n).
\end{aligned}$$

The quotient space  $\mathbb{CP}^n/T^n$  is known to be homeomorphic to  $\Delta^n$  and the stratification  $\pi_{\mathbb{CP}^n/T^n}$  can be identified with the stratification  $\pi_n$  on  $\Delta^n$  in Example 2.11. This stratification, however, does not induce the minimal cell decomposition of  $\mathbb{CP}^n$ . The other stratification  $\pi_n^{\max}$  on  $\Delta^n$  defined in Example 2.11 induces the minimal cell decomposition on  $\mathbb{CP}^n$  by the composition

$$\mathbb{CP}^n \xrightarrow{p} \mathbb{CP}^n/T^n \cong \Delta^n \xrightarrow{\pi_n^{\max}} [n].$$

Let us show that this cell decomposition is cylindrically normal. To this end, we first rewrite  $\mathbb{CP}^n$  by using the construction introduced in [DJ91] by Davis and Januszkiewicz. Given a simple polytope<sup>19</sup>  $P$  of dimension  $n$  and a function  $\lambda : \{\text{codimension-1 faces in } P\} \rightarrow \mathbb{C}^n$  satisfying certain conditions, they constructed a space  $M(\lambda)$  with  $T^n$ -action. Suppose  $P = \Delta^n$  and define

$$\lambda_n(C_i) = \begin{cases} (1, \dots, 1), & i = 0 \\ (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i}), & i = 1, \dots, n, \end{cases}$$

where  $C_i$  is the codimension-1 face with vertices in  $[n] - \{i\}$ . In this case,  $M(\lambda_n)$  can be described as

$$M(\lambda_n) = (T^n \times \Delta^n) / \sim,$$

where the equivalence relation  $\sim$  is generated by the following relations: Let  $p = (p_0, \dots, p_n) \in \Delta^n$ .

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<sup>19</sup>A  $d$ -dimensional convex polytope is said to be *simple* if each vertex is adjacent to exactly  $d$ -edges.

- When  $p_i = 0$  for  $1 \leq i \leq n$ ,

$$(t_1, \dots, t_i, \dots, t_n; p_0, \dots, p_n) \sim (t_1, \dots, t'_i, \dots, t_n; p_0, \dots, p_n)$$

for any  $t_i, t'_i \in S^1$ .

- When  $p_0 = 0$ ,

$$(t_1, \dots, t_n; 0, p_1, \dots, p_n) \sim (\omega t_1, \dots, \omega t_n; 0, p_1, \dots, p_n)$$

for any  $\omega \in S^1$ .

An explicit homeomorphism  $p_n : \mathbb{CP}^n \rightarrow M(\lambda_n)$  and its inverse  $q_n$  are given by

$$\begin{aligned} p_n([z_0 : \dots : z_n]) &= \begin{cases} \left[ \frac{z_1/z_0}{|z_1/z_0|}, \dots, \frac{z_n/z_0}{|z_n/z_0|}; \frac{\sum_{i=0}^n |z_i|^2}{\sum_{i=0}^n |z_i|^2}, \dots, \frac{\sum_{i=0}^n |z_n|^2}{\sum_{i=0}^n |z_i|^2} \right] & z_0 \neq 0 \\ \left[ \frac{z_1}{|z_1|}, \dots, \frac{z_n}{|z_n|}; 0, \frac{\sum_{i=0}^n |z_1|^2}{\sum_{i=0}^n |z_i|^2}, \dots, \frac{\sum_{i=0}^n |z_n|^2}{\sum_{i=0}^n |z_i|^2} \right] & z_0 = 0 \end{cases} \\ q_n([z_1, \dots, z_n; x_0, \dots, x_n]) &= [\sqrt{x_0}, \sqrt{x_1} z_1, \dots, \sqrt{x_n} z_n]. \end{aligned}$$

Under this identification, the minimal cell decomposition on  $\mathbb{CP}^n$  can be described as

$$\mathbb{CP}^n \cong M(\lambda_n) = \bigcup_{i=1}^n (T^n / T^{n-i} \times (\Delta^i - \Delta^{i-1})) / \sim.$$

Regard  $D^{2n} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 \leq 1\}$  and define a map

$$\varphi_{2n} : D^{2n} \longrightarrow M(\lambda_n)$$

by

$$\varphi_{2n}(z_1, \dots, z_n) = \left[ \frac{z_1}{|z_1|}, \dots, \frac{z_n}{|z_n|}; 1 - \sum_{i=1}^n |z_i|^2, |z_1|^2, \dots, |z_n|^2 \right].$$

This is a cell structure map for the  $2n$ -cell. For  $m < n$ , define

$$b_{2m,2n} : S^1 \times D^{2m} \longrightarrow D^{2n}$$

by

$$b_{2m,2n}(\omega, z_1, \dots, z_m) = \left( 0, \dots, 0, \omega \sqrt{1 - \sum_{i=1}^m |z_i|^2}, \omega z_1, \dots, \omega z_m \right).$$

Then each  $b_{2m,2n}|_{S^1 \times \text{Int}(D^{2m})}$  is a homeomorphism onto its image and we have a stratification

$$\partial D^{2n} = S^{2n-1} = \bigcup_{m=0}^{n-1} b_{2m,2n}(S^1 \times \text{Int}(D^{2m})).$$

Furthermore the diagram

$$\begin{array}{ccc} & D^{2n} & \xrightarrow{\varphi_{2n}} M(\lambda_n) \\ & \uparrow & \uparrow \\ S^1 \times D^{2m} & \xrightarrow{b_{2m,2n}} S^{2n-1} & \\ \text{pr}_2 \downarrow & & \\ D^{2m} & \xrightarrow{\varphi_{2m}} M(\lambda_m), & \end{array}$$

is commutative, where the inclusion  $M(\lambda_m) \hookrightarrow M(\lambda_n)$  is given by

$$[t_1, \dots, t_m; p_0, \dots, p_m] \mapsto [1, \dots, 1, t_1, \dots, t_m; 0, \dots, 0, p_0, \dots, p_m].$$

Now define  $P_{2m,2n} = S^1$  for  $m < n$ . The group structure of  $S^1$  defines a map

$$c_{2\ell,2m,2n} : P_{2m,2n} \times P_{2\ell,2n} \longrightarrow P_{2\ell,2n}$$

making the diagram

$$\begin{array}{ccc} S^1 \times S^1 \times D^{2\ell} & \xrightarrow{1 \times b_{2\ell,2m}} & S^1 \times D^{2m} \\ c_{2\ell,2m,2n} \times 1 \downarrow & & \downarrow b_{2m,2n} \\ S^1 \times D^{2\ell} & \xrightarrow{b_{2\ell,2n}} & D^{2n} \end{array}$$

commutative and we have a cylindrical structure.

More generally, Davis and Januszkiewicz [DJ91] proved that any quasitoric manifold  $M$  of dimension  $2n$  can be expressed as  $M \cong M(\lambda)$  for a simple convex polytope  $P$  of dimension  $n$  and a function  $\lambda$ . The right hand side is a space constructed as a quotient of  $T^n \times P$  under an equivalence relation analogous to the case of  $\mathbb{CP}^n$ . Davis and Januszkiewicz proved in §3 of their paper that there is a “perfect Morse function” on  $M(\lambda)$  which induces a cell decomposition of  $M(\lambda)$  or  $M$  with exactly  $h_i(P)$  cells of dimension  $2i$ , where  $(h_0(P), \dots, h_n(P))$  is the  $h$ -vector of  $P$ . It seems very likely that the above construction of a cylindrical structure on  $\mathbb{CP}^n$  can be extended to quasitoric manifolds.  $\square$

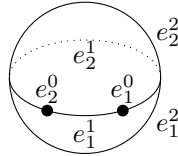
**Example 3.28.** In the same paper, Davis and Januszkiewicz introduced the notion of small covers as a real analogue of quasitoric manifolds by replacing  $S^1$  by  $\mathbb{Z}_2$ . Small covers have many properties in common with quasitoric (or torus) manifolds.

For example, we have  $\mathbb{RP}^n/(\mathbb{Z}_2)^n \cong \Delta^n$  and the stratification  $\pi_n^{\max}$  on  $\Delta^n$  induces the minimal cell decomposition of  $\mathbb{RP}^n$ . An argument analogous to the case of  $\mathbb{CP}^n$  can be used to prove that this stratification is totally normal.  $\square$

**Example 3.29.** Let  $X = D^3$  and consider the cell decomposition

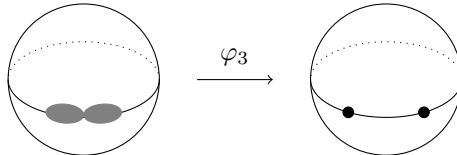
$$X = e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1 \cup e_1^2 \cup e_2^2 \cup e^3$$

given as follows.



Namely the interior of  $D^3$  is the unique 3-cell and the boundary  $S^2$  is cut into two 2-cells by the equator, which is cut into two 1-cells by two 0-cells on it.

Consider the map between  $S^2$  given by collapsing the shaded region in the figure below (the wedge of two 2-disks embedded in  $S^2$ ) “vertically” and expanding the remaining part of  $S^2$  continuously. Extend it to a continuous map  $\varphi_3 : D^3 \rightarrow D^3$ .



It defines a 3-cell structure on  $e^3$ . Let  $P_{(1,1),3} = D^1$ . We have a continuous map

$$b : P_{(1,1),3} \times D^1 \longrightarrow \partial D^3$$

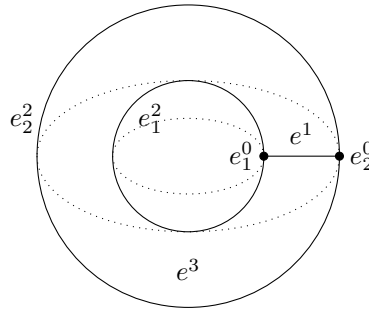
making the diagram

$$\begin{array}{ccc} D^3 & \xrightarrow{\varphi_3} & X \\ \uparrow b & & \uparrow \varphi_{1,1} \\ P_{(1,1),3} \times D^1 & \longrightarrow & D^1 \end{array}$$

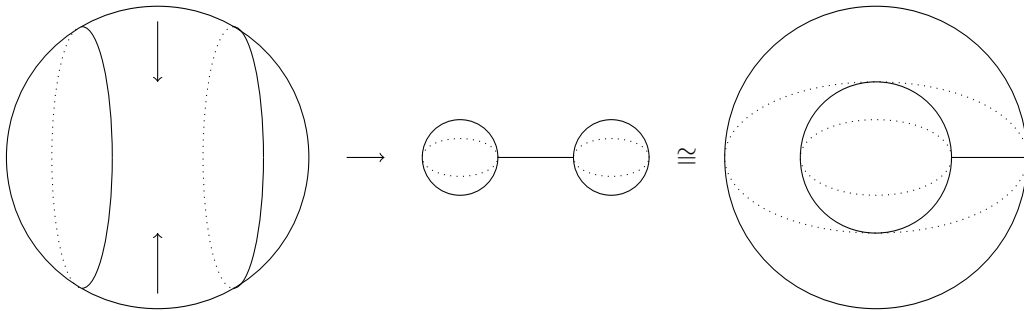
commutative. However,  $b$  is not a homeomorphism when restricted to  $P_{(1,1),3} \times \text{Int}(D^1)$ . And this is not a cylindrical structure.  $\square$

In Examples 3.25 and Example 3.26, the restrictions of cell structure maps to the boundary spheres are fiber bundles onto their images. These facts seem to be closely related to the existence of cylindrical structures in these examples. Of course, there are many cylindrically normal cellular stratified spaces that do not have such bundle structures. We may be able to characterize cylindrically normal cellular stratified spaces by using an appropriate notion of stratified fiber bundles. See [Dav78] for example.

**Example 3.30.** Let  $X$  be the subspace of the unit 3-disk  $D^3$  obtained by removing the interior of the 3-disk of radius  $\frac{1}{2}$  centered at the origin. It has a cell decomposition with two 0-cells, a 1-cell, two 2-cells, and a 3-cell depicted as follows.



Cells of dimension at most 1 are regular. The cell structure maps for 2-cells are given as in Example 3.20. The restriction of the cell structure map  $\varphi_3$  for the 3-cell to  $S^2$  is given by collapsing a middle part of a sphere to a segment.



This cell decomposition is cylindrically normal, because of the “local triviality” of the middle band. However the restriction  $\varphi_3|_{S^2}$  is not a fiber bundle onto its image.  $\square$

**Example 3.31.** We have seen in Example 3.17 that the standard cell decomposition of the geometric realization of a  $\Delta$ -set is totally normal. Let us consider the geometric realization of a simplicial set  $X$ . Define

$$P(X) = \coprod_{n=0}^{\infty} \left( X_n - \bigcup_{i=0}^n s_i(X_{n-1}) \right)$$

to be the set of nondegenerate simplices. For  $\sigma, \tau \in P(X)$ , define  $\tau \leq \sigma$  if there exists an injective morphism  $u : [m] \rightarrow [n]$  with  $X(u)(\sigma) = \tau$ . Define

$$\pi_X : |X| \longrightarrow P(X)$$

analogously to the case of  $\Delta$ -sets. Then  $\pi_X$  is a cellular stratification. In other words, cells in  $|X|$  are in one-to-one correspondence to nondegenerate simplices.

Suppose  $\tau \leq \sigma$  in  $P(X)$ . Then the set

$$\mathcal{P}(\tau, \sigma) = \{u : [m] \rightarrow [n] \mid u \text{ injective and } X(u)(\sigma) = \tau\}$$

is nonempty. For  $u, v \in \mathcal{P}(\tau, \sigma)$ , define  $u \leq v$  if and only if  $u(i) \leq v(i)$  for all  $i \in [m]$ . Let us denote the order complex of this poset by  $P_{\tau, \sigma}$ . This is a simplicial complex whose simplices are indexed by chains in  $\mathcal{P}(\tau, \sigma)$  and can be written as

$$P_{\tau, \sigma} = \bigcup_{k=0}^{\infty} \bigcup_{\mathbf{u} \in N_k(\mathcal{P}(\tau, \sigma))} \Delta^k \times \{\mathbf{u}\}.$$

For a  $k$ -chain  $\mathbf{u} = \{u_0 < \dots < u_k\} \in N_k(\mathcal{P}(\tau, \sigma))$ , define a map

$$\beta^{\mathbf{u}} : [k] \times [m] \longrightarrow [n]$$

by

$$\beta^{\mathbf{u}}(i, j) = u_i(j).$$

This map induces an affine map

$$b^{\mathbf{u}} : (\Delta^k \times \{\mathbf{u}\}) \times \Delta^m \longrightarrow \Delta^n.$$

These maps can be glued together to give us a map

$$b_{\tau, \sigma} : P_{\tau, \sigma} \times \Delta^m \longrightarrow \Delta^n.$$

For  $\sigma_0 < \sigma_1 < \sigma_2$  in  $P(X)$ , the composition

$$\mathcal{P}(\sigma_1, \sigma_2) \times \mathcal{P}(\sigma_0, \sigma_1) \longrightarrow \mathcal{P}(\sigma_0, \sigma_2)$$

is a morphism of posets and induces a map

$$c_{\sigma_0, \sigma_1, \sigma_2} : P_{\sigma_1, \sigma_2} \times P_{\sigma_0, \sigma_1} \longrightarrow P_{\sigma_0, \sigma_2}.$$

It is straightforward to check that these maps satisfy the requirements of a cylindrical structure. Thus the standard cell decomposition of the geometric realization  $|X|$  is cylindrically normal.  $\square$

We require morphisms between cylindrically normal cellular or stellar stratified spaces to preserve cylindrical structures.

**Definition 3.32.** Let  $(X, \pi_X, \Phi_X)$  and  $(Y, \pi_Y, \Phi_Y)$  be cylindrically normal stellar stratified spaces with cylindrical structures given by  $\{b_{\mu,\lambda}^X : P_{\mu,\lambda}^X \times D_\mu \rightarrow D_\lambda\}$  and  $\{b_{\alpha,\beta}^Y : P_{\alpha,\beta}^Y \times D_\alpha \rightarrow D_\beta\}$ , respectively.

A *morphism of cylindrically normal stellar stratified spaces* from  $(X, \pi_X, \Phi_X)$  to  $(Y, \pi_Y, \Phi_Y)$  is a morphism of stellar stratified spaces

$$f = (f, \underline{f}) : (X, \pi_X, \Phi_X) \longrightarrow (Y, \pi_Y, \Phi_Y)$$

together with maps  $f_{\mu,\lambda} : P_{\mu,\lambda}^X \rightarrow P_{\underline{f}(\mu), \underline{f}(\lambda)}^Y$  making the diagram

$$\begin{array}{ccc} P_{\mu,\lambda}^X \times D_\mu & \xrightarrow{f_{\mu,\lambda} \times f_\mu} & P_{\underline{f}(\mu), \underline{f}(\lambda)}^Y \times D_{\underline{f}(\mu)} \\ \downarrow b_{\mu,\lambda}^X & & \downarrow b_{\underline{f}(\mu), \underline{f}(\lambda)}^Y \\ D_\lambda & \xrightarrow{f_\lambda} & D_{\underline{f}(\lambda)} \end{array}$$

commutative.

The categories of cylindrically normal cellular and stellar stratified spaces are denoted by  $\mathbf{CSSpaces}^{\text{cyl}}$  and  $\mathbf{SSSpaces}^{\text{cyl}}$ , respectively.

### 3.3 Locally Polyhedral Cellular Stratified Spaces

Although total normality and cylindrical normality play central roles in our study of cellular and stellar stratified spaces, it is not easy to prove a given cellular stratification is totally normal or cylindrically normal.

In order to address this problem, we found polyhedral complexes and PL maps are useful in [BGRT] and introduced a structure called “locally polyhedral CW complex”. It turns out that PL maps also play an important role in our proof of Theorem 4.16. In this section, we generalize locally polyhedral CW complexes and introduce the notion of locally polyhedral cellular stratified spaces.

Let us first recall the definition of polyhedral complexes. See §2.2.4 of Kozlov’s book [Koz08] for details.

**Definition 3.33.** A *Euclidean polyhedral complex* is a subspace  $K$  of  $\mathbb{R}^N$  for some  $N$  equipped with a finite family of maps

$$\{\varphi_i : F_i \longrightarrow K \mid i = 1, \dots, n\}$$

satisfying the conditions that

1. each  $F_i$  is a convex polytope;
2. each  $\varphi_i$  is an affine equivalence onto its image;
3.  $K = \bigcup_{i=1}^n \varphi_i(F_i)$ ;
4. for  $i \neq j$ ,  $\varphi_i(F_i) \cap \varphi_j(F_j)$  is a proper face of  $\varphi_i(F_i)$  and  $\varphi_j(F_j)$ .

The polytopes  $F_i$ ’s are called *generating polytopes*.

Obviously a polyhedral complex is a regular cell complex. By replacing affine cell structure maps  $\varphi_i$  by continuous maps, Kozlov defined a more general kind of polyhedral complexes in his book. In order to be even more general, we introduced “locally polyhedral CW complexes” in



[BGRT] as a cell decomposition on a subspace of a Euclidean space whose cells are Euclidean polyhedral complexes and whose cell structure maps are PL. The requirement of being a subspace of  $\mathbb{R}^N$  can be removed if we assume cylindrical normality. We may also remove the condition that all cells are closed. Here is our new definition of “locally polyhedral” structure.

**Definition 3.34.** A *locally polyhedral stellar stratified space* consists of

- a CW cylindrically normal stellar stratified space  $X$ ,
- a family of Euclidean polyhedral complexes  $\tilde{F}_\lambda$  indexed by  $\lambda \in P(X)$  and
- a family of homeomorphisms  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow \overline{D_\lambda}$  indexed by  $\lambda \in P(X)$ , where  $\overline{D_\lambda}$  is the closure of the domain stellar cell  $D_\lambda$  for  $e_\lambda$  in a disk containing  $D_\lambda$ ,

satisfying the following conditions:

1. For each cell  $e_\lambda$ ,  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow \overline{D_\lambda}$  is a subdivision of stratified space, where the stratification on  $\overline{D_\lambda}$  is defined by the cylindrical structure<sup>20</sup>.
2. For each pair  $e_\mu < e_\lambda$ , the parameter space  $P_{\mu,\lambda}$  is a locally cone-like space<sup>21</sup> and the composition

$$P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda$$

is a PL map<sup>22</sup>, where  $F_\lambda = \alpha_\lambda^{-1}(D_\lambda)$ .

Each  $\alpha_\lambda$  is called a *polyhedral replacement* of the cell structure map of  $e_\lambda$ . The collection  $A = \{\alpha_\lambda\}_{\lambda \in P(X)}$  is called a *locally-polyhedral structure* on  $X$ .

**Remark 3.35.** When  $X$  is cellular,  $\overline{D_\lambda} = D^{\dim e_\lambda}$ .

**Remark 3.36.** Besides locally polyhedral CW complexes in [BGRT], an analogous structure for cell complexes is introduced by A. Kirillov, Jr. in [Kir] as PLCW complexes for a different purpose.  $M_\kappa$ -polyhedral complexes in the book [BH99] by Bridson and Haefliger are also closely related.

**Definition 3.37.** A *morphism of locally polyhedral stellar stratified spaces* from  $(X, \pi, \Phi, A)$  to  $(X', \pi', \Phi', A')$  consists of a morphism of stellar stratified spaces

$$(f, \{f_\lambda\}) : (X, \pi, \Phi) \longrightarrow (X', \pi', \Phi')$$

and a family of PL maps  $\tilde{f}_\lambda : \tilde{F}_\lambda \rightarrow \tilde{F}'_{f(\lambda)}$  for  $\lambda \in P(X)$  that are compatible with locally-polyhedral structures.

Any polyhedral complex is locally polyhedral. More generally, any “locally polyhedral cylindrically normal CW complex” in the sense of [BGRT] has a local polyhedral structure in the above sense.

**Lemma 3.38.** *Let  $X$  be a subspace of  $\mathbb{R}^N$  equipped with a structure of cylindrically normal CW cellular stratified space whose parameter spaces  $P_{\mu,\lambda}$  are locally cone-like spaces. Suppose, for*

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<sup>20</sup>Definition 3.23

<sup>21</sup>Definition A.27

<sup>22</sup>Definition A.36

each  $e_\lambda \in P(X)$ , there exists a polyhedral complex  $\tilde{F}_\lambda$  and a homeomorphism  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$  such that the composition

$$P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\varphi_\lambda} X \hookrightarrow \mathbb{R}^N$$

is a PL map, where  $F_\lambda = \alpha_\lambda^{-1}(D_\lambda)$ . Suppose further that  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$  is a cellular subdivision, where the cell decomposition on  $D^{\dim e_\lambda}$  is the one in the definition of cylindrical structure. Then the collection  $\{\alpha_\lambda\}_{\lambda \in P(X)}$  defines a local polyhedral structure on  $X$ .

*Proof.* For each pair  $e_\mu < e_\lambda$ , define  $\tilde{b}_{\mu,\lambda} : P_{\mu,\lambda} \times F_\mu \rightarrow F_\lambda$  to be the composition

$$\tilde{b}_{\mu,\lambda} : P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda.$$

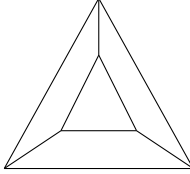
Note that, when  $\mu = \lambda$ ,  $b_{\mu,\lambda}$  is the identity map. Thus the top horizontal composition in the following diagram is a PL map:

$$\begin{array}{ccccccc} F_\lambda & \xrightarrow{\alpha_\lambda} & D_\lambda & \xrightarrow{\varphi_\lambda} & X & \hookrightarrow & \mathbb{R}^N \\ \tilde{b}_{\mu,\lambda} \uparrow & & & & & & \parallel \\ P_{\mu,\lambda} \times F_\mu & \xrightarrow{1 \times \alpha_\mu} & P_{\mu,\lambda} \times D_\mu & \xrightarrow{b_{\mu,\lambda}} & D_\lambda & \xrightarrow{\varphi_\lambda} & X \hookrightarrow \mathbb{R}^N. \end{array}$$

The bottom horizontal composition is also a PL map by assumption and  $\tilde{b}_{\mu,\lambda}$  is an embedding when restricted to  $P_{\mu,\lambda} \times \alpha_\mu^{-1}(\text{Int } D^{\dim e_\mu})$ . Thus  $\tilde{b}_{\mu,\lambda}$  is also PL by Lemma A.38.  $\square$

The following example is discussed in [BGRT].

**Example 3.39.** Consider the minimal cell decomposition  $S^2 = e^0 \cup e^2$ . We have an embedding  $f : S^2 \rightarrow \mathbb{R}^3$  whose image is the boundary  $\partial\Delta^3$  of the standard 3-simplex and  $f(e^0)$  is a vertex of  $\partial\Delta^3$ . This is a cylindrically normal cellular stratified space by Example 3.25. Let  $P$  be a 2-dimensional polyhedral complex in  $\mathbb{R}^2$  described by the following figure



By collapsing the outer triangle, we obtain a map

$$\psi : P \longrightarrow \partial\Delta^3$$

whose restriction to the interior is a homeomorphism. Let  $\alpha : P \rightarrow D^2$  be a homeomorphism given by a radial expansion. Then maps  $f$  and  $\varphi$  can be chosen in such a way they make the following diagram commutative

$$\begin{array}{ccc} D^2 & \xrightarrow{\varphi} & S^2 \\ \alpha \uparrow & & \downarrow f \\ P & \xrightarrow{\psi} & \partial\Delta^3 \hookrightarrow \mathbb{R}^3. \end{array}$$

By Lemma 3.38, we obtain a structure of locally polyhedral cellular stratified space on  $S^2$  or  $\partial\Delta^3$ .  $\square$

**Definition 3.40.** A cellular stratified space satisfying the assumption of Lemma 3.38 are called a *Euclidean locally polyhedral cellular stratified space*.

Totally normal cellular stratified spaces form an important class of locally polyhedral cellular stratified spaces.

**Lemma 3.41.** *Any CW totally normal cellular stratified space has a locally polyhedral structure.*

*Proof.* Let  $X$  be a CW totally normal cellular stratified space. By definition, for each cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  in  $X$ , there exists a regular cell decomposition of  $D^{\dim e_\lambda}$  containing  $D_\lambda$  as a cellular stratified subspace. Since the barycentric subdivision of a regular cell complex has a structure of simplicial complex,  $D^{\dim e_\lambda}$  can be embedded in a Euclidean space as a finite simplicial complex  $\tilde{F}_\lambda$ . By induction on dimensions of cells, we may choose homeomorphisms  $\{\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}\}_{\lambda \in P(X)}$  in such a way the composition

$$F_\mu \xrightarrow{\alpha_\mu} D_\mu \xrightarrow{b} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda$$

is a PL map for each lift  $b : D_\mu \rightarrow D_\lambda$  of the cell structure map of  $e_\mu$  for each pair  $e_\mu < e_\lambda$ . Thus we obtain a locally polyhedral structure.  $\square$

Conversely, the local polyhedrality condition provides us with a useful criterion for a stratification to be totally normal. The following fact is proved in [BGRT].

**Theorem 3.42.** *Let  $X$  be a normal CW complex embedded in  $\mathbb{R}^N$  for some  $N$ . Suppose for each cell  $e \subset X$  with cell structure map  $\varphi$ , there exists a Euclidean polyhedral complex  $F$  and a homeomorphism  $\alpha : F \rightarrow D^{\dim e}$  such that the composition*

$$F \xrightarrow{\alpha} D^{\dim e} \xrightarrow{\varphi} X \subset \mathbb{R}^N$$

*is a PL map. Then any regular cellular stratified subspace of  $X$  is totally normal.*

*Proof.* Theorem 6.1.8 in [BGRT].  $\square$

**Example 3.43.** The product cell decomposition of  $(S^n)^k$  induced by the minimal cell decomposition on  $S^n$  is locally polyhedral. In [BGRT], a subdivision of the product cell decomposition was defined by using the stratification on  $(\mathbb{R}^n)^\ell$  associated with the braid arrangement  $\mathcal{A}_{\ell-1}$  for  $1 \leq \ell \leq k$ .

The resulting stratification, called the *braid stratification*, is locally polyhedral and contains the configuration space  $\text{Conf}_k(S^n)$  as a stratified subspace. It turns out the induced stratification on  $\text{Conf}_k(S^n)$  is regular and hence it is totally normal.  $\square$

**Remark 3.44.** We study products, subdivisions, and subspaces of cellular stratified spaces in a forthcoming paper [Tama].

The above example suggests the following strategy to prove a given cellular stratification on  $X$  is cylindrically normal or totally normal:

1. Embed  $X$  into a cylindrically normal or totally normal cell complex  $\tilde{X}$ .
2. Find an appropriate subdivision of  $\tilde{X}$  which includes  $X$  as a stratified subspace.

In fact, it is easy to prove that, if  $X$  is locally polyhedral,  $X$  can be embedded in a cell complex as a stratified subspace.

**Lemma 3.45.** *Let  $X$  be a locally polyhedral cellular stratified space. Given a pair of cells  $e_\mu \subset \partial e_\lambda$ , the structure map*

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow D_\lambda$$

*has a unique extension to the whole disks*

$$\overline{b_{\mu,\lambda}} : P_{\mu,\lambda} \times D^{\dim e_\mu} \longrightarrow D^{\dim e_\lambda}.$$

*Proof.* By Lemma A.38. □

These maps allow us to construct a canonical closure of any locally polyhedral cellular stratified space.

**Definition 3.46.** Let  $X$  be a locally polyhedral cellular stratified space with cells  $\{e_\lambda\}_{\lambda \in P(X)}$ . Define a cell complex  $U(X)$  by

$$U(X) = \left( \coprod_{\lambda \in P(X)} D^{\dim e_\lambda} \right) / \sim$$

where the relation  $\sim$  is the equivalence relation generated by the following relation: For  $x \in D^{\dim e_\lambda}$  and  $y \in D^{\dim e_\mu}$ ,  $x \sim y$  if  $e_\mu \subset \partial e_\lambda$  and there exists  $z \in P_{\mu,\lambda}$  such that  $\overline{b_{\mu,\lambda}}(z, y) = x$ .

There is a canonical inclusion

$$i : X \hookrightarrow U(X).$$

This space  $U(X)$  is called the *cellular closure* of  $X$ .

By definition, we have the following.

**Lemma 3.47.** *When  $X$  is an locally polyhedral cellular stratified space,  $U(X)$  is a cylindrically normal CW complex containing  $X$  as a cellular stratified subspace.*

**Example 3.48.** In the case of the punctured torus in Example 3.16,  $U(X)$  is obtained by gluing parallel edges in  $I^2$  and is homeomorphic to  $T^2$ . In other words,  $U(X)$  is obtained by closing the hole in  $X$ . □

## 4 Topological Face Categories and Their Classifying Spaces

Recall that the collection of all cells in a regular cell complex  $X$  forms a poset whose order complex is homeomorphic to  $X$ . We also have a face poset for any cellular stratified space. One of the main results in [BGRT] is that the order complex of the face poset of a regular totally normal cellular stratified space  $X$  can be embedded in  $X$  as a strong deformation retract.

For non-regular cellular stratified spaces, however, we cannot expect to recover the homotopy type of the original space from its face poset. We should construct an acyclic topological category from cells. See Appendix B for basics of topological categories.

## 4.1 Face Categories

There are several ways to construct a category from a cellular or stellar stratified space. A naive idea is to use the family of sets  $\{F(X)(e_\mu, e_\lambda)\}_{\mu, \lambda \in P(X)}$  defined in Definition 3.6 as morphism sets.

**Definition 4.1.** For a cellular or a stellar stratified space  $(X, \pi, \Phi)$ , define a topological category  $F(X, \pi, \Phi)$  as follows. Objects are cells

$$F(X, \pi, \Phi)_0 = \{e \mid \text{cells in } (X, \pi)\}.$$

$F(X, \pi, \Phi)_0$  is equipped with the discrete topology.

A morphism from a cell  $\varphi : D \rightarrow \bar{e}$  to another cell  $\varphi' : D' \rightarrow \bar{e}'$  is a lift of the cell structure map  $\varphi$  of  $e$ , i.e. a map  $b : D \rightarrow D'$  making the following diagram commutative:

$$\begin{array}{ccc} D' & \xrightarrow{\varphi'} & \bar{e}' \hookrightarrow X \\ \uparrow b & & \nearrow \\ D & \xrightarrow{\varphi} & \bar{e}. \end{array}$$

Note that the existence of a morphism  $b : e \rightarrow e'$  implies  $\bar{e} \subset \bar{e}'$ . The set of morphisms  $F(X, \pi, \Phi)(e, e')$  from  $e$  to  $e'$  is topologized by the compact-open topology as a subspace of  $\text{Map}(D, D')$ . The composition is given by the composition of maps.

This topological category  $F(X, \pi, \Phi)$  is called the *face category* of  $(X, \pi, \Phi)$ . It is denoted by  $F(X, \pi)$  or  $F(X)$ , when  $\pi$  or  $\Phi$  is obvious from the context.

**Lemma 4.2.**  $F(X)$  is an acyclic category. When  $X$  is regular,  $F(X)$  is a poset and coincides with the face poset  $P(X)$ . In particular, when  $X$  is a regular cell complex, our construction coincides with the classical face poset construction.

*Proof.* When  $\dim e > \dim e'$ ,  $e' \subset \partial e$  and the only possible direction of morphisms is  $e' \rightarrow e$ . When  $\dim e = \dim e'$ , the compatibility of lifts with cell structure maps implies that the only case we have a morphism is  $e = e'$  and the morphism should be the identity. Thus  $F(X)$  is acyclic.

When  $X$  is regular, the regularity implies that there is at most one morphism between two objects. Hence it is a poset.  $\square$

Even when  $X$  is not regular, we have the underlying poset<sup>23</sup>, since  $F(X)$  is an acyclic category. Obviously it is isomorphic to the face poset  $P(X, \pi) = \text{Im } \pi$ .

**Lemma 4.3.** For a strongly normal<sup>24</sup> cellular or stellar stratified space  $X$ , the underlying poset  $P(F(X))$  coincides with the face poset  $P(X)$ .

**Example 4.4.** Consider the minimal cell decomposition

$$\pi_n : S^n = e^0 \cup e^n \longrightarrow \{0 < n\}.$$

The face category  $F(S^n, \pi_n)$  has two objects  $e^0$  and  $e^n$ . When  $n = 1$ , as we have seen in Example 3.14,

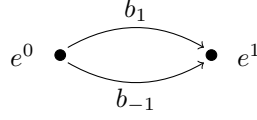
$$F(S^1, \pi_1)(e^0, e^1) = \{b_1, b_{-1}\}.$$

We also have the identity morphism on each object. The resulting category can be depicted as follows:

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<sup>23</sup>Definition B.6

<sup>24</sup>Definition 3.6



When  $n > 1$ , there are infinitely many morphisms from  $e^0$  to  $e^n$  parametrized by  $\partial D^n$ . And we have a homeomorphism

$$F(S^n, \pi_n)(e^0, e^n) \cong S^{n-1}.$$

□

In the above example of  $S^n$ , the compact-open topology on the morphism space  $F(S^n, \pi_n)(e^0, e^n)$  can be replaced with a more understandable topology of  $S^{n-1}$ . In general, we cannot expect such simplicity. Under the assumption of cylindrical normality, however, we may define a smaller face category.

**Definition 4.5.** Let  $X$  be a cylindrically normal stellar stratified space. Define a category  $C(X)$  as follows. Objects are cells in  $\pi$ . For each pair  $e_\mu \subset \overline{e_\lambda}$ , define

$$C(X)(e_\mu, e_\lambda) = P_{\mu, \lambda}.$$

The composition of morphisms is given by

$$c_{\lambda_0, \lambda_1, \lambda_2} : P_{\lambda_1, \lambda_2} \times P_{\lambda_0, \lambda_1} \longrightarrow P_{\lambda_0, \lambda_2}.$$

The category  $C(X)$  is called the *cylindrical face category* of  $X$ .

**Lemma 4.6.** For a cylindrically normal stellar stratified space  $X$ , the maps  $b_{\mu, \lambda}$  induces a continuous functor

$$b : C(X) \longrightarrow F(X),$$

which is natural with respect to morphisms of cylindrically normal stellar stratified spaces.

Furthermore the underlying poset of  $C(X)$  is also  $P(X)$  and the diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{b} & F(X) \\ & \searrow & \swarrow \\ & P(X) & \end{array}$$

is commutative.

*Proof.* The continuity of

$$b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \longrightarrow D_\lambda$$

implies the continuity of its adjoint

$$\text{ad}(b_{\mu, \lambda}) : P_{\mu, \lambda} \longrightarrow \text{Map}(D_\mu, D_\lambda),$$

which factors through  $F(X)(e_\mu, e_\lambda)$ . It is immediate to verify that these maps form a continuous functor

$$b : C(X) \longrightarrow F(X).$$

□

**Example 4.7.** Consider the minimal cell decomposition on  $\mathbb{CP}^2$

$$\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4.$$

It is shown in Example 3.26 that it has a cylindrical structure. The cylindrical face category  $C(\mathbb{CP}^2)$  has three objects,  $e^0$ ,  $e^2$ , and  $e^4$ . We have seen

$$F(\mathbb{CP}^2)(e^0, e^2) = F(S^2, \pi_2)(e^0, e^2) \cong S^1 = C(\mathbb{CP}^2)(e^0, e^2).$$

Since the attaching map of  $e^4$  is the Hopf map

$$\eta : S^3 \longrightarrow \mathbb{CP}^1,$$

we have

$$F(\mathbb{CP}^2)(e^0, e^4) \cong \eta^{-1}(e^0) \cong S^1 = C(\mathbb{CP}^2)(e^0, e^4).$$

By using the local trivialization

$$\eta^{-1}(e^2) \cong e^2 \times S^1,$$

we see that  $F(\mathbb{CP}^2)(e^2, e^4)$  is the set of sections of the trivial bundle

$$D^2 \times S^1 \longrightarrow D^2$$

and thus  $F(\mathbb{CP}^2)(e^2, e^4) = \text{Map}(D^2, S^1)$ . On the other hand, we have

$$C(\mathbb{CP}^2)(e^2, e^4) = S^1$$

by definition. The composition

$$C(\mathbb{CP}^2)(e^2, e^4) \times C(\mathbb{CP}^2)(e^0, e^2) \longrightarrow C(\mathbb{CP}^2)(e^0, e^4)$$

is given by the multiplication of  $S^1$ , as is shown in Example 3.26.

In general, Example 3.27 says that the face category  $C(\mathbb{CP}^n)$  of the minimal cell decomposition of  $\mathbb{CP}^n$  can be described as a “poset enriched by  $S^1$ ”

$$\begin{array}{ccccccc} & S^1 & & S^1 & & & S^1 \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet \end{array}$$

in the sense that, for any pair of objects  $e^{2k}, e^{2m}$  ( $k < m$ ), the space of morphisms  $C(\mathbb{CP}^n)(e^{2k}, e^{2m})$  is  $S^1$  and the composition of morphisms is given by the group structure of  $S^1$ .  $\square$

Recall that the order complex of the face poset of a regular cell complex  $X$  is the barycentric subdivision of  $X$ . With this fact in mind, we introduce the following notation.

**Definition 4.8.** Let  $X$  be a cylindrically normal stellar stratified space. Define its *barycentric subdivision*  $\text{Sd}(X)$  to be the classifying space of the cylindrical face category

$$\text{Sd}(X) = BC(X).$$

**Remark 4.9.** There is a notion of barycentric subdivision  $\text{Sd}(C)$  of a small category  $C$ . A good reference is a paper [dH08] by del Hoyo. See also Noguchi’s papers [Nog11, Nog]. We show that, for a totally normal stellar stratified space  $X$ , there is an isomorphism of categories  $\text{Sd}(C(X)) \cong C(\text{Sd}(X))$  in [Tama].

When  $X$  is not a regular cell complex, we might not have a homeomorphism between  $\text{Sd}(X)$  and  $X$ .

**Example 4.10.** Consider  $X = \mathbb{R}^n$ . This is a regular totally normal cellular stratification consisting of a single  $n$ -cell. The barycentric subdivision is a single point.  $\square$

**Example 4.11.** Consider the minimal cell decomposition  $\pi_n$  of  $S^n$ . When  $n = 1$ , it is easy to see that  $\text{Sd}(S^1, \pi_1)$  is the following cell complex and is homeomorphic to  $S^1$ :



Note that this complex can be regarded as the barycentric subdivision of  $\pi_1$ .

When  $n > 1$ ,  $C(S^n, \pi_n)$  is a topological category with nontrivial topology on  $C(S^n, \pi_n)(e^0, e^n)$ . Since we have a homeomorphism

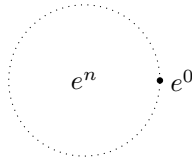
$$C(S^n, \pi_n)(e^0, e^n) \cong S^{n-1},$$

it is easy to determine  $\text{Sd}(S^n, \pi_n)$  and we have

$$\text{Sd}(S^n, \pi_n) = BC(S^n, \pi_n) \cong \Sigma(S^{n-1}) \cong S^n.$$

Again we recovered  $S^n$ .  $\square$

**Example 4.12.** Consider  $X = \text{Int}D^n \cup \{(1, 0)\}$  with the obvious stratification.

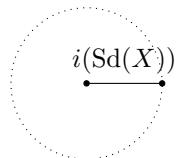


This is a regular cellular stratification and  $\text{Sd}(X)$  is a 1-simplex  $[0, 1]$ . We have an embedding

$$i : [0, 1] \longrightarrow X$$

by

$$i(t) = (1 - t)(1, 0) + t(0, 0).$$



Obviously  $i([0, 1])$  is a strong deformation retract of  $X$ .  $\square$

**Example 4.13.** Consider the punctured torus in Example 3.16. There is a totally normal cellular stratification on  $X = S^1 \times S^1 - e^0 \times e^0$  induced from the product cell decomposition  $\pi_1^2$

$$S^1 \times S^1 = e^0 \times e^0 \cup e^0 \times e^1 \cup e^1 \times e^0 \cup e^1 \times e^1.$$

Let

$$\varphi_{1,1} : D_{1,1} = [-1, 1]^2 - \{(-1, -1), (-1, 1), (1, -1), (1, 1)\} \longrightarrow X$$

be the cell structure map of the 2-cell in  $X$  and

$$\begin{aligned} \varphi_{0,1} & : D_{0,1} = (-1, 1) \longrightarrow X \\ \varphi_{1,0} & : D_{1,0} = (-1, 1) \longrightarrow X \end{aligned}$$



be the cell structure maps for 1-cells.

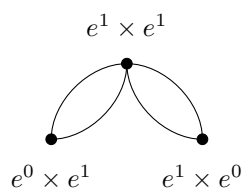
As we have seen in Example 3.16, there are two ways to lift each cell structure map of a 1-cell and these four lifts cover  $\partial D_{1,1}$ .

$$\partial D_{1,1} = b'_{0,1}(D_{0,1}) \cup b''_{0,1}(D_{0,1}) \cup b'_{1,0}(D_{1,0}) \cup b''_{1,0}(D_{1,0}).$$

The cylindrical face category  $C(X)$  consists of three objects  $e^0 \times e^1$ ,  $e^1 \times e^0$ ,  $e^1 \times e^1$ . Nontrivial morphisms are

$$\begin{aligned} C(X)(e^0 \times e^1, e^1 \times e^1) &= \{b'_{0,1}, b''_{0,1}\}, \\ C(X)(e^1 \times e^0, e^1 \times e^1) &= \{b'_{1,0}, b''_{1,0}\}. \end{aligned}$$

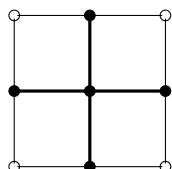
By allowing multiple edges, we can draw a “Hasse diagram” of this acyclic category as follows.



Obviously, the classifying space of this category is the wedge of two circles,

$$\mathrm{Sd}(X) = BC(X) = S^1 \vee S^1.$$

It is easy to find an embedding of  $\mathrm{Sd}(X)$  into  $X$  by using lifts of cell structure maps. The images of 0 under the four maps  $b'_{0,1}$ ,  $b''_{0,1}$ ,  $b'_{1,0}$ ,  $b''_{1,0}$  constitute four points in  $\partial D_{1,1}$  that are mapped to a single point under  $\varphi_{1,1}$ . By connecting each of these four points and  $(0,0)$  in  $D_{1,1}$  by a segment, respectively, we obtain a 1-dimensional stratified space  $\tilde{K}$  in  $D_{1,1}$ .



The thick part  $\tilde{K}$  in the above picture corresponds to cells in  $\mathrm{Sd}(X)$  and we have an embedding

$$\mathrm{Sd}(X) \hookrightarrow X.$$

The above picture can be also used to construct a deformation retraction of  $X$  onto  $\mathrm{Sd}(X)$ .  $\square$

The above examples show that, when a cellular stratification  $\pi$  is totally normal, or more generally, cylindrically normal, the barycentric subdivision  $\mathrm{Sd}(X, \pi)$  is closely related to  $X$ .

The work of Cohen, Jones, and Segal [CJS] suggests that one analyze the nerve of the face category of a cylindrically normal stellar stratified space by using the underlying poset functor

$$\pi : C(X, \pi) \longrightarrow P(X, \pi).$$

The following easily verifiable fact will be used in the next section.

**Lemma 4.14.** *For a cylindrically normal stellar stratified space  $(X, \pi)$ , consider the induced morphism of simplicial sets<sup>25</sup>*

$$N(\pi) : N(C(X, \pi)) \longrightarrow N(P(X, \pi)).$$

*For each  $n$ -chain in the face poset  $\mathbf{e} = (e_{\lambda_0}, \dots, e_{\lambda_n}) \in N_n(P(X, \pi))$ , we have*

$$N(\pi)_n^{-1}(\mathbf{e}) = P_{\lambda_{n-1}, \lambda_n} \times \dots \times P_{\lambda_0, \lambda_1}.$$

*Consequently the space of  $n$ -chains has the following decomposition*

$$N_n(C(X, \pi)) = \coprod_{\mathbf{e} \in N_n(P(X, \pi))} \{\mathbf{e}\} \times P_{\lambda_{n-1}, \lambda_n} \times \dots \times P_{\lambda_0, \lambda_1}.$$

*The space of nondegenerate  $n$ -chains is, therefore, given by*

$$\overline{N}_n(C(X, \pi)) = \coprod_{\mathbf{e} \in \overline{N}_n(P(X, \pi))} \{\mathbf{e}\} \times P_{\lambda_{n-1}, \lambda_n} \times \dots \times P_{\lambda_0, \lambda_1}.$$

## 4.2 Barycentric Subdivisions of Cellular Stratified Spaces

As we can see from the examples in §4.1, we often have an embedding of the barycentric subdivision  $\text{Sd}(X)$  in  $X$ , even if  $X$  is neither a cell complex nor regular. In this section, we first show that such an embedding always exists for any cylindrically normal cellular stratified space. And then we show that it often admits a strong deformation retraction.

In order to describe our embedding, we use the language of simplicial spaces. For a topological space  $X$ , the simplicial space<sup>26</sup> defined by  $S_n(X) = \text{Map}(\Delta^n, X)$  with the compact-open topology is denoted by  $S(X)$ .

**Theorem 4.15.** *Let  $(X, \pi)$  be a cylindrically normal CW stellar stratified space. Then there exists a morphism of simplicial spaces*

$$i : N(C(X, \pi)) \longrightarrow S(X),$$

*such that the composition*

$$\tilde{i} : \text{Sd}(X, \pi) = |N(C(X, \pi))| \xrightarrow{|i|} |S(X)| \xrightarrow{\text{ev}} X$$

*is an embedding. The map  $\tilde{i}$  is natural with respect to strict morphisms of cylindrically normal stellar stratified spaces. Furthermore, when all cells in  $X$  are closed,  $\tilde{i}$  is a homeomorphism.*

*Proof.* By the (local) compactness of  $\Delta^n$ , defining a map of simplicial spaces

$$i : N(C(X, \pi)) \longrightarrow S(X)$$

is equivalent to constructing a family of maps

$$i_n : N_n(C(X, \pi)) \times \Delta^n \longrightarrow X$$

<sup>25</sup>Here we forget the topology on  $C(X, \pi)$  temporarily.

<sup>26</sup>See Example A.11.

making the diagrams

$$\begin{array}{ccccc}
& & N_n(C(X, \pi)) \times \Delta^n & & \\
& \nearrow^{1 \times d^i} & & \searrow^{i_n} & \\
N_n(C(X, \pi)) \times \Delta^{n-1} & & & & X \\
& \searrow_{d_i \times 1} & & \nearrow_{i_{n-1}} & \\
& & N_{n-1}(C(X, \pi)) \times \Delta^{n-1} & &
\end{array}$$
  

$$\begin{array}{ccccc}
& & N_n(C(X, \pi)) \times \Delta^n & & \\
& \nearrow^{s_i \times 1} & & \searrow^{i_n} & \\
N_{n-1}(C(X, \pi)) \times \Delta^n & & & & X \\
& \searrow_{1 \times s^i} & & \nearrow_{i_{n-1}} & \\
& & N_{n-1}(C(X, \pi)) \times \Delta^{n-1} & &
\end{array}$$

commutative for all  $i$ .

Let us construct  $i_n$  by induction on  $n$ . The set  $N_0(C(X, \pi))$  is the set of cells in  $\pi$ . For each cell  $e_\lambda$ , let

$$\varphi_\lambda : D_\lambda \longrightarrow \overline{e_\lambda} \subset X$$

be the cell structure map of  $e_\lambda$  and define

$$i_0(e_\lambda) = \varphi_\lambda(0)$$

and we obtain

$$i_0 : N_0(C(X, \pi)) \cong N_0(C(X, \pi)) \times \Delta^0 \hookrightarrow X.$$

Note that this map makes the diagram

$$\begin{array}{ccc}
N_0(C(X, \pi)) \times \Delta^0 & \xrightarrow{i_0} & X \\
\downarrow z_0 & \nearrow \Phi & \\
\coprod_{\lambda} \{e_\lambda\} \times D_\lambda & &
\end{array}$$

commutative, where  $z_0$  is induced by the inclusion of  $\Delta^0$  to the center of each disk and  $\Phi$  is given by the cell structure maps. Let us denote

$$\Delta(X) = \coprod_{\lambda} \{e_\lambda\} \times D_\lambda$$

for simplicity.

Suppose that we have constructed a map

$$i_{k-1} : N_{k-1}(C(X)) \times \Delta^{k-1} \longrightarrow X$$

satisfying the above compatibility conditions and that the restriction of  $i_{k-1}$  to  $N_{k-1}(C(X)) \times \text{Int}(\Delta^{k-1})$  is an embedding. Suppose, further, that there exists a map

$$z_{k-1} : N_{k-1}(C(X)) \times \Delta^{k-1} \longrightarrow \Delta(X)$$

making the diagram

$$\begin{array}{ccc}
 N_{k-1}(C(X)) \times \Delta^{k-1} & \xrightarrow{i_{k-1}} & X \\
 \downarrow z_{k-1} & \nearrow \Phi & \\
 \Delta(X) & & 
 \end{array}$$

commutative. We construct

$$z_k : N_k(C(X)) \times \Delta^k \longrightarrow \Delta(X)$$

satisfying the compatibility conditions corresponding to those of  $i_k$  and define  $i_k$  to be  $\Phi \circ z_k$ .

Under the decomposition in Lemma 4.14

$$N_k(C(X)) = \coprod_{e \in N_k(P(X))} \{e\} \times N(\pi)_k^{-1}(e),$$

it suffices to construct a map

$$z_e : N(\pi)_k^{-1}(e) \times \Delta^k \longrightarrow D_{\lambda_k}$$

for each  $k$ -chain  $e : e_{\lambda_0} \leq \dots \leq e_{\lambda_k}$  in  $P(X)$ . The maps  $\{z_e\}$  should satisfy the following conditions:

1. For each  $0 \leq j < k$ , the following diagram is commutative:

$$\begin{array}{ccc}
 N(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D_{\lambda_k} \\
 \uparrow 1 \times d^j & & \uparrow z_{d_j(e)} \\
 N(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{d_j \times 1} & N(\pi)_{k-1}^{-1}(d_j(e)) \times \Delta^{k-1}.
 \end{array} \tag{2}$$

2. When  $j = k$ , the following diagram is commutative:

$$\begin{array}{ccc}
 N(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D_{\lambda_k} \\
 \uparrow 1 \times d^k & & \uparrow b_{\lambda_{k-1}, \lambda_k} \\
 N(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xlongequal{\quad} P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} \xrightarrow{1 \times z_{d_k(e)}} P_{\lambda_{k-1}, \lambda_k} \times D_{\lambda_{k-1}}.
 \end{array}$$

3. For each  $0 \leq j \leq k$ , the following diagram is commutative:

$$\begin{array}{ccc}
 N(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D_{\lambda_k} \\
 \uparrow 1 \times s^j & & \uparrow z_{s_j(e)} \\
 N(\pi)_k^{-1}(e) \times \Delta^{k+1} & \xrightarrow{s_j \times 1} & N(\pi)_{k+1}^{-1}(s_j(e)) \times \Delta^{k+1}.
 \end{array}$$

The third condition implies that it suffices to construct  $z_e$  for nondegenerate  $k$ -chains  $e : e_{\lambda_0} < e_{\lambda_1} < \dots < e_{\lambda_k}$  in  $P(X)$ . By the inductive assumption, we have a map

$$z_{d_k(e)} : N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} \longrightarrow D_{\lambda_{k-1}}$$

corresponding to the  $(k-1)$ -chain  $d_k(e) : e_{\lambda_0} < \dots < e_{\lambda_{k-1}}$ . Note that

$$N(\pi)_k^{-1}(e) = P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)).$$

Compose with  $b_{\lambda_{k-1}, \lambda_k}$  and we obtain a map

$$\begin{aligned} N(\pi)_k^{-1}(e) \times \Delta^{k-1} &= P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} \\ &\xrightarrow{1 \times z_{d_k(e)}} P_{\lambda_{k-1}, \lambda_k} \times D_{\lambda_{k-1}} \\ &\xrightarrow{b_{\lambda_{k-1}, \lambda_k}} \partial D_{\lambda_k}. \end{aligned}$$

Since  $D_{\lambda_k}$  is an aster<sup>27</sup>, it can be extended to a map

$$z_e : N(\pi)_k^{-1}(e) \times \Delta^k = N(\pi)_k^{-1}(e) \times \Delta^{k-1} * e_k \longrightarrow \partial D_{\lambda_k} * 0 \subset D_{\lambda_k}$$

by

$$z_e(\mathbf{p}, (1-t)\mathbf{s} + t\mathbf{v}_k) = (1-t)z_{d_k(e)}(\mathbf{p}, \mathbf{s}) + t \cdot 0 = (1-t)z_{d_k(e)}(\mathbf{p}, \mathbf{s}),$$

where  $\mathbf{v}_k = (0, \dots, 0, 1)$  is the last vertex in  $\Delta^k$ . Recall that the join operation  $*$  used in this paper is defined by connecting points by line segments<sup>28</sup> and is not the same as the join operation used in algebraic topology, for example in Milnor's paper [Mil56].

It remains to show that  $z_e$  makes the diagram (2) commutative or  $0 \leq j < k$ . By the inductive assumption, the following diagram is commutative:

$$\begin{array}{ccc} N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-2} & \xrightarrow{d_j \times 1} & N(\pi)_{k-2}^{-1}(d_j d_k(e)) \times \Delta^{k-2} \quad \equiv \quad N(\pi)_{k-2}^{-1}(d_{k-1} d_j(e)) \times \Delta^{k-2} \\ \downarrow 1 \times d^j & & \downarrow z_{d_j d_k(e)} \\ N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} & \xrightarrow{z_{d_k(e)}} & D_{\lambda_{k-1}} \end{array}$$

$\swarrow z_{d_{k-1} d_j(e)}$

Under the identification  $\Delta^k = \Delta^{k-1} * \mathbf{v}_k$ ,  $d^j : \Delta^{k-1} \rightarrow \Delta^k$  can be identified with the composition

$$\Delta^{k-2} * \mathbf{v}_{k-1} \cong \Delta^{k-2} * \mathbf{v}_k \xrightarrow{d^j * 1_{\mathbf{v}_k}} \Delta^{k-1} * \mathbf{v}_k.$$

On the other hand, since  $j < k$ , we have

$$\begin{aligned} N(\pi)_k(e) &= P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \\ N(\pi)_{k-1}(d_j(e)) &= P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-2}^{-1}(d_{k-1} d_j(e)), \end{aligned}$$

and the face operator  $d_j : N(\pi)_k(e) \rightarrow N(\pi)_{k-1}(d_j(e))$  coincides with the map

$$P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \xrightarrow{1 \times d_j} P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_j d_k(e)) = P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_{k-1} d_j(e)).$$

---

<sup>27</sup>Definition 2.54

<sup>28</sup>Definition A.25.

Thus we obtain the commutative diagram

$$\begin{array}{ccc}
N(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{d_j \times 1} & N(\pi)_{k-1}^{-1}(d_j(e)) \times \Delta^{k-1} \\
\parallel & & \parallel \\
P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} & \xrightarrow{1 \times d_j \times 1} & P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-2}^{-1}(d_{k-1}d_j(e)) \times \Delta^{k-1} \\
\downarrow 1 \times 1 \times d^j & & \downarrow 1 \times \tilde{z}_{d_{k-1}d_j(e)} \\
P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^k & \xrightarrow{1 \times \tilde{z}_{d_k(e)}} & P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1))
\end{array}$$

where we embed  $D_{\lambda_{k-1}}$  into  $D_{\lambda_{k-1}} \times \mathbb{R}$  as  $D_{\lambda_{k-1}} \times \{0\}$  and  $\tilde{z}_{d_k(e)}$  is defined by

$$\tilde{z}_{d_k(e)}(\mathbf{p}, (1-t)\mathbf{s} + t\mathbf{v}_k) = (1-t)(z_{d_k(e)}(\mathbf{p}, \mathbf{s}), 0) + t(0, 1).$$

The map  $\tilde{z}_{d_{k-1}d_j(e)}$  is defined analogously. We also have an extension of  $b_{\lambda_{k-1}, \lambda_k}$

$$\tilde{b}_{\lambda_{k-1}, \lambda_k} : P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1)) \longrightarrow D_{\lambda_k}.$$

By definition, we have the commutative diagram

$$\begin{array}{ccc}
P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-2}^{-1}(d_{k-1}d_j(e)) \times \Delta^{k-1} & \xlongequal{\quad} & N(\pi)_{k-1}^{-1}(d_j(e)) \times \Delta^{k-1} \\
\downarrow 1 \times \tilde{z}_{d_{k-1}d_j(e)} & & \downarrow z_{d_j(e)} \\
P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1)) & \xrightarrow{\tilde{b}_{\lambda_{k-1}, \lambda_k}} & D_{\lambda_k} \\
\uparrow 1 \times \tilde{z}_{d_k(e)} & & \uparrow z_e \\
P_{\lambda_{k-1}, \lambda_k} \times N(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^k & \xlongequal{\quad} & N(\pi)_k^{-1}(e) \times \Delta^k.
\end{array}$$

This completes the inductive construction of  $z_k$  and we obtain maps

$$i_k : N_k(C(X)) \times \Delta^k \longrightarrow X$$

and a map

$$\tilde{i} : \text{Sd}(X) = |N(C(X))| \longrightarrow X.$$

By the definition of cylindrical structure, the restriction  $b_{\lambda_{k-1}, \lambda_k}|_{P_{\lambda_{k-1}, \lambda_k} \times \text{Int} D_{\lambda_{k-1}}}$  is an embedding. It implies the restriction  $i_k|_{N_k(C(X)) \times \text{Int}(\Delta^k)} : N_k(C(X)) \times \text{Int}(\Delta^k) \rightarrow X$  is an embedding for each  $k$  and  $\tilde{i}$  is a continuous monomorphism. The fact that it is an open map follows from the commutativity of the diagram

$$\begin{array}{ccc}
\coprod_k N_k(C(X)) \times \Delta^k & \xrightarrow{\coprod_k i_k} & \Delta(X) \\
\downarrow & & \downarrow \Phi \\
|N(C(X))| & \xrightarrow{\tilde{i}} & X
\end{array}$$

whose vertical arrows are quotient maps. When all cells are closed, each  $z_k$  is surjective and thus  $\tilde{i}$  is a homeomorphism.  $\square$

In [BGRT], we proved that, when  $(X, \pi)$  is totally normal and regular, the above map  $\tilde{i}$  embeds  $\text{Sd}(X, \pi)$  as a deformation retract of  $X$ . The following is an extension of this result. Note that we need locally polyhedral structures.

**Theorem 4.16.** *For a locally polyhedral cellular stratified space  $X$ , the image of the above map  $\tilde{i} : \text{Sd}(X, \pi) \rightarrow X$  is a strong deformation retract of  $X$ . The deformation retraction can be taken to be natural with respect to strict morphisms of cylindrically normal cellular stratified spaces.*

In particular, we may remove the regularity assumption in Theorem 6.2.4 in [BGRT].

**Corollary 4.17.** *For a totally normal cellular stratified space  $(X, \pi)$ , the map  $\tilde{i}$  embeds  $\text{Sd}(X, \pi)$  into  $X$  as a strong deformation retract.*

We need the following fact proved in [BGRT].

**Lemma 4.18.** *Let  $\pi$  be a regular cell decomposition of  $S^{n-1}$  and  $L \subset S^{n-1}$  be a stratified subspace. Let  $\tilde{\pi}$  be the cellular stratification on  $K = \text{Int}D^n \cup L$  obtained by adding  $\text{Int}D^n$  as an  $n$ -cell. Then there is a deformation retraction  $H$  of  $K$  to  $\tilde{i}(\text{Sd}(K, \tilde{\pi}))$ . Furthermore if a deformation retraction  $h$  of  $L$  onto  $i(\text{Sd}(L))$  is given,  $H$  can be taken to be an extension of  $h$ .*

This Lemma can be proved by using good old simplicial topology. For the convenience of the reader, a proof is included in Appendix §A.1.

Now we are ready to prove Theorem 4.16.

*Proof of Theorem 4.16.* Let us show the embedding  $\tilde{i}$  constructed in the proof of Theorem 4.15 has a homotopy inverse.

Since  $X$  is locally polyhedral, each cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  has a polyhedral replacement  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$ . For simplicity, we identify  $D_\lambda$  with  $\alpha_\lambda^{-1}(D_\lambda)$ . Thus  $D_\lambda$  is a stratified subspace of a polyhedral complex and  $b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \rightarrow D_\lambda$  is a PL map.

We construct, by induction on  $k$ , a PL homotopy

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

for each  $k$ -cell  $e_\lambda$  satisfying the following conditions:

1. It is a strong deformation retraction of  $D_\lambda$  onto  $\tilde{i}(\text{Sd}(D_\lambda))$ , where the stratification on  $D_\lambda$  is given by adding  $\text{Int}(D_\lambda)$  as a unique  $k$  cell to the stratification on  $\partial D_\lambda$ .
2. The diagram

$$\begin{array}{ccc} D_\lambda \times [0, 1] & \xrightarrow{H_\lambda} & D_\lambda \\ \uparrow b_{\mu, \lambda} \times 1 & & \uparrow b_{\mu, \lambda} \\ P_{\mu, \lambda} \times D_\mu \times [0, 1] & \xrightarrow{1 \times H_\mu} & P_{\mu, \lambda} \times D_\mu. \end{array} \quad (3)$$

is commutative for any pair  $e_\mu \subset \overline{e_\lambda}$ .

When  $k = 0$ , the homotopy is the canonical projection. Suppose we have constructed  $H_\mu$  for all  $i$ -cells  $e_\mu$  with  $i \leq k - 1$ . We would like to extend them to  $k$ -cells. For a  $k$ -cell  $e_\lambda$  with cell structure map  $\varphi_\lambda : D_\lambda \rightarrow X$  and a cell  $e_\mu$  with  $e_\mu \subset \overline{e_\lambda}$ , consider the diagram

$$\begin{array}{ccc} P_{\mu,\lambda} \times D_\mu \times [0, 1] & \xrightarrow{1 \times H_\mu} & P_{\mu,\lambda} \times D_\mu \\ \downarrow b_{\mu,\lambda} \times 1 & & \downarrow b_{\mu,\lambda} \\ b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu) \times [0, 1] & \cdots \cdots \cdots \rightarrow & b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu). \end{array}$$

Since  $b_{\mu,\lambda}$  is a homeomorphism when restricted to  $P_{\mu,\lambda} \times \text{Int}(D_\mu)$  and  $b_{\mu,\lambda}$  and  $H_\mu$  are assumed to be PL, we have the dotted arrow making the diagram commutative by Lemma A.38.

The decomposition

$$\partial D_\lambda = \bigcup_{e_\mu \subset \overline{e_\lambda}} b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu)$$

allows us to glue these homotopies together. Thus we obtain a homotopy

$$H_\lambda : \partial D_\lambda \times [0, 1] \longrightarrow \partial D_\lambda.$$

By Lemma 4.18, this homotopy can be extended to a strong deformation retraction

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

of  $D_\lambda$  onto  $\tilde{i}(\text{Sd}(D_\lambda))$ . This completes the inductive step.

The second condition above (3) implies that these deformation retractions can be assembled together to give a strong deformation retraction

$$H : X \times [0, 1] \longrightarrow X$$

of  $X$  onto  $\tilde{i}(\text{Sd}(X))$ . □

As we have seen in [BGR], the construction of the higher order Salvetti complex can be regarded as a corollary to Theorem 4.16.

**Example 4.19.** Consider the stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell} : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \text{Map}(L, S_\ell)$$

in Example 2.10. As we have seen in Example 2.47, this is a normal cellular stratification. It can be easily seen to be regular as follows. Let  $B_r^{n\ell}$  be the closed ball centered at the origin with radius  $r$  in  $\mathbb{R}^n \otimes \mathbb{R}^\ell$ . The stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  induces a regular cell decomposition on  $B_r^{n\ell}$ . By taking  $r$  to be large enough, we may identify the face poset  $P(\mathcal{A} \otimes \mathbb{R}^\ell)$  with the face poset of the cellular stratification on  $\text{Int} B_r^{n\ell}$ . Thus  $(\mathbb{R}^n \otimes \mathbb{R}^\ell, \mathcal{A} \otimes \mathbb{R}^\ell)$  is a totally normal regular cellular stratification. It is also locally polyhedral.

It contains

$$\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell) = \bigcup_{i=1}^k H_i \otimes \mathbb{R}^\ell$$

as a cellular stratified subspace. The complement

$$M(\mathcal{A} \otimes \mathbb{R}^\ell) = \mathbb{R}^n \otimes \mathbb{R}^\ell - \text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)$$



is also a cellular stratified subspace. Since  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  is regular, we have

$$C(M(\mathcal{A} \otimes \mathbb{R}^\ell)) = P(M(\mathcal{A} \otimes \mathbb{R}^\ell)) = P(\mathcal{A} \otimes \mathbb{R}^\ell) - P(\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)).$$

By Theorem 4.16,  $BC(M(\mathcal{A} \otimes \mathbb{R}^\ell))$  can be embedded in the complement  $M(\mathcal{A} \otimes \mathbb{R}^\ell)$  as a strong deformation retract. This simplicial complex  $BC(M(\mathcal{A} \otimes \mathbb{R}^\ell))$  is nothing but the higher order Salvetti complex in [BZ92, DCS00].  $\square$

The above example is within a framework developed in [BGRT]. The next example is totally normal but not regular.

**Example 4.20.** By Example 3.15, a graph  $X$  can be regarded as a totally normal cellular stratified space. As we is described in [FT], we may define a totally normal cellular stratification on  $X^k$  by using the braid arrangements, which can be restricted to a totally normal cellular stratification on  $\text{Conf}_k(X)$ . By Corollary 4.17,  $BC(\text{Conf}_k(X))$  can be embedded in  $\text{Conf}_k(X)$  as a  $\Sigma_k$ -equivariant strong deformation retract.

$BC(\text{Conf}_k(X))$  is a regular CW complex model for  $\text{Conf}_k(X)$ . This is better than Abrams' model in the sense that it works for all graphs.  $\square$

The cellular stratification on  $\mathbb{CP}^n$  in Example 3.27 is cylindrically normal and Theorem 4.15 applies.

**Example 4.21.** Consider the cylindrically normal cellular stratification on  $\mathbb{CP}^n$  in Example 3.26 and Example 3.27. By Theorem 4.15,  $BC(\mathbb{CP}^n)$  is homeomorphic to  $\mathbb{CP}^n$ . Let us compare the description of  $BC(\mathbb{CP}^n)$  with the one in Example 3.27 as the Davis-Januszkiewicz construction  $M(\lambda_n)$ .

As we have seen in Example 4.7, the cylindrical face category  $C(\mathbb{CP}^n)$  can be obtained from the underlying poset  $[n]$  of  $C(\mathbb{CP}^n)$  by replacing the set of morphisms by  $S^1$ . Compositions of morphisms are given by the group structure of  $S^1$ .

By Lemma 4.14, we have

$$\overline{N}_k(C(\mathbb{CP}^n)) = \coprod_{\mathbf{e} \in \overline{N}_k([n])} \{\mathbf{e}\} \times (S^1)^k$$

and the classifying space of  $C(\mathbb{CP}^n)$  can be described in the form

$$\begin{aligned} BC(\mathbb{CP}^n) &= \|\overline{N}(C(\mathbb{CP}^n))\| \\ &= \left( \coprod_k \coprod_{\mathbf{e} \in \overline{N}_k([n])} \{\mathbf{e}\} \times (S^1)^k \times \Delta^k \right) / \sim. \end{aligned}$$

Note that a nondegenerate  $k$ -chain  $\mathbf{e}$  in the face poset of  $\Delta^n$  can be describe by a strictly increasing sequence of nonnegative integers  $\mathbf{e} = (i_0, \dots, i_k)$  with  $i_k \leq n$ . For an element  $(\mathbf{e}; t_1, \dots, t_k; p_0, \dots, p_k)$  in  $\overline{N}_k(C(\mathbb{CP}^n))$ , let  $\tilde{\mathbf{t}}$  be an element of  $(S^1)^n$  obtained from  $(t_1, \dots, t_k)$  by inserting 1 in such a way that each  $t_j$  is placed between  $i_{j-1}$ -th and  $i_j$ -th positions. Then there exists a face operator  $d_I : N_n(C(\mathbb{CP}^n)) \rightarrow N_k(C(\mathbb{CP}^n))$  such that

$$(\mathbf{e}; t_1, \dots, t_k) = (d_I(0, 1, \dots, n), d_I(\tilde{\mathbf{t}})).$$

And we have

$$(\mathbf{e}; t_1, \dots, t_k; p_0, \dots, p_k) \sim (0, \dots, n; \tilde{\mathbf{t}}; d^I(p_0, \dots, p_k)).$$

Note that  $d^I(p_0, \dots, p_k)$  is obtained from  $(p_0, \dots, p_k)$  by inserting 0 in appropriate coordinates. Thus any point in  $BC(\mathbb{CP}^n)$  can be represented by a point in  $(S^1)^n \times \Delta^n$  and  $BC(\mathbb{CP}^n)$  can be written as

$$BC(\mathbb{CP}^n) = (S^1)^n \times \Delta^n / \sim. \quad (4)$$

The relation here is not exactly the same as the defining relation of  $M(\lambda_n)$ .

Define a map

$$s_n : M(\lambda_n) \longrightarrow BC(\mathbb{CP}^n)$$

by

$$s_n([t_1, \dots, t_n; p_0, \dots, p_n]) = [t_1, t_1^{-1}t_2, \dots, t_{n-1}^{-1}t_n; p_0, \dots, p_n].$$

It is left to the reader to verify that  $s_n$  is a well-defined homeomorphism making the diagram

$$\begin{array}{ccc} M(\lambda_n) & \xrightarrow{s_n} & BC(\mathbb{CP}^n) \\ \downarrow & & \downarrow B\pi_{C(\mathbb{CP}^n)} \\ \Delta^n & \xlongequal{\quad} B([n]) \xlongequal{\quad} & BP(\mathbb{CP}^n) \end{array}$$

commutative, where  $\pi_{C(\mathbb{CP}^n)} : C(\mathbb{CP}^n) \rightarrow P(\mathbb{CP}^n)$  is the canonical projection onto the underlying poset<sup>29</sup>. Thus we see that  $BC(\mathbb{CP}^n)$  coincides with  $M(\lambda_n)$  up to a homeomorphism.  $\square$

## 5 Concluding Remarks

Before we conclude this paper, we would like to discuss (possible) applications of cellular stratified spaces, including open problems.

### 5.1 Problems on Cellular Stratified Spaces in General

We need to understand products, subdivisions, and subspaces of cellular stratified spaces in order to apply the results in this paper to configuration spaces. We study these operations in [Tama].

Other than these operations, an extension of Theorem 4.16 to stellar stratified spaces would be important. Although Theorem 4.15 holds for stellar stratified spaces, Theorem 4.16 is restricted to cellular stratified spaces, since it depends on Lemma 4.18. It should not be difficult to prove a stellar analogue of Theorem 4.16, once we have a stellar version of Lemma 4.18.

**Problem 5.1.** Find an appropriate condition on stellar stratified spaces under which an analogue of Theorem 4.16 holds.

Another important difference between Theorem 4.15 and Theorem 4.16 is that we need locally-polyhedral (PL) structures in the latter. This requirement is also due to Lemma 4.18.

**Problem 5.2.** Replace the locally-polyhedral assumption in Theorem 4.16 by a more general topological condition, such as the existence of NDR structures.

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<sup>29</sup>Definition B.6

## 5.2 Combinatorial Models of Configuration Spaces and Complements of Arrangements

Our primary motivation for this work is to build a common framework for working with configuration spaces and complements of arrangements. We already succeeded in constructing a combinatorial model for  $\text{Conf}_k(S^n)$  in [BGRT]. We should be able to extend the construction to other configuration spaces. A particularly easy case is the configuration spaces of graphs. We have seen in Example 4.20 that, for a graph  $X$ , the cellular stratifications of Euclidean spaces by the braid arrangements can be extended to a totally normal cellular subdivision of the product stratification  $X^k$  which contains the configuration space  $\text{Conf}_k(X)$  as a cellular stratified subspace. By Theorem 4.16, the barycentric subdivision  $\text{Sd}(\text{Conf}_k(X)) = BC(\text{Conf}_k(X))$  serves as a good cellular model for  $\text{Conf}_k(X)$ . For example, Ghrist's theorem in [Ghr01] that, for a finite graph  $X$ , the homotopy dimension of  $\text{Conf}_k(X)$  is bounded by the number of essential vertices can be proved easily by using our model. In fact, the dimension of our model  $\text{Sd}(\text{Conf}_k(X))$  is exactly the number of essential vertices when we choose the minimal cell decomposition of  $X$ . This result will appear soon as a joint work with Mizuki Furuse [FT].

One of Salvetti's motivations for his work [Sal87] on the complement of the complexification of a real hyperplane arrangement  $\mathcal{A}$  was to study the fundamental group  $\pi_1(M(\mathcal{A} \otimes \mathbb{C}))$  of the complement. He constructed a regular cell complex  $\text{Sal}(\mathcal{A})$  whose barycentric subdivision is  $BC(M(\mathcal{A} \otimes \mathbb{C})) = \text{Sd}(M(\mathcal{A} \otimes \mathbb{C}))$ . The fundamental group of an unordered configuration space of a graph  $X$ , i.e. the graph braid group of  $X$ , has been studied by using Abrams' cubical model for configuration spaces of graphs by many authors. Thus we should be able to get a good hold on graph braid groups by using the face category  $F(\text{Conf}_k(X)) = C(\text{Conf}_k(X))$ .

**Problem 5.3.** Given a graph  $X$ , find a presentation of  $\pi_1(\text{Conf}_k(X)/\Sigma_k)$  in terms of the combinatorial structure of  $X$  or  $F(\text{Conf}_k(X))$ .

The notion of cellular stratified space was introduced in [BGRT] in order to study a numerical invariant of topological spaces, called the higher symmetric topological complexity. We obtained a good estimate of  $\text{TC}_k^S(S^n)$  by studying the homotopy dimensions of the unordered configuration spaces  $\text{Conf}_m(S^n)/\Sigma_m$  for  $m \leq k$ . It would be very useful if we could extend the construction of the braid stratification of  $\text{Conf}_k(S^n)$  to other cell complexes, although it seems difficult in general. The case of real projective spaces seems to be tractable. It should be noted, however, that we need to use a finer stratification than the braid stratification.

**Problem 5.4.** Extend the construction of the braid stratification on  $\text{Conf}_k(S^n)$  to  $\text{Conf}_k(\mathbb{RP}^n)$  and  $\text{Conf}_k(\mathbb{CP}^n)$  and study their properties.

**Problem 5.5.** Given a locally polyhedral cellular stratified space  $X$ , find a systematic way to subdivide the product stratification  $X^k$  in such a way that the resulting stratification contains the discriminant set  $\Delta_k(X)$  as a cellular stratified subspace.

If we were to have such a stratification on  $X^k$ , we would obtain a cellular model for  $\text{Conf}_k(X)$ . When  $X$  is a finite cell complex, such a model would be a compact model for  $\text{Conf}_k(X)$ . The following question was asked by Robert Ghrist during the International Symposium on Nonlinear Theory and its Applications 2011.

**Problem 5.6.** Let  $X$  be a closed manifold. When  $X^k$  has a subdivision which includes  $\Delta_k(X)$  as a cellular stratified subspace, find an explicit relation between the combinatorial structures of the cellular model obtained from the theory of cellular stratified spaces and the Axelrod-Singer-Fulton-MacPherson compactification of  $\text{Conf}_k(X)$ .

We should be able to apply Theorem 4.16 to obtain “Salvetti-type complexes” for other types of configuration spaces and arrangements. For example, Moci and Settepanella [MS] constructed a version of Salvetti complex for toric arrangements. The construction was generalized by d’Antonio and Delucchi [dD] by using classifying spaces of acyclic categories. It seems reasonable to expect that their face category of a complexified toric arrangement comes from a totally normal cellular stratification.

### 5.3 Morse Theory

Our definition of cylindrical structures is inspired by the work of Cohen, Jones, and Segal [CJS, CJS94, CJS95] on Morse theory. Given a Morse-Smale function  $f : M \rightarrow \mathbb{R}$  on a closed manifold  $M$ , we have a cellular decomposition of  $M$  by unstable manifolds. It is reasonable to expect that this Morse-theoretic cell decomposition is cylindrically normal and the resulting face category  $C(M)$  coincides with the topological category  $C(f)$  constructed in [CJS]. We should be able to extend their work<sup>30</sup> to non-closed manifolds such as configuration spaces by using results of the current work.

**Problem 5.7.** Extend the discussion in [CJS] by Cohen, Jones, and Segal to open manifolds, manifolds with boundaries, and manifolds with corners by using the notion of cellular stratified spaces.

Note that we required the existence of a locally polyhedral structure in Theorem 4.16. Our proof depends heavily on PL-topological techniques. In the context of Cohen-Jones-Segal Morse theory, we should replace the PL-topological arguments by differential-topological ones.

**Problem 5.8.** Define a smooth version of locally polyhedral cellular stratified space and prove an analogue of Theorem 4.16.

**Problem 5.9.** Define a notion of smooth cellular stratified spaces in such a way that

1. a Morse-Smale function  $f : M \rightarrow \mathbb{R}$  on a smooth manifold defines a smooth cellular stratification on  $M$ ,
2. the barycentric subdivision  $\text{Sd}(X)$  of a smooth cellular stratified space  $X$  carries a structure of smooth manifold with singularity, and
3. the Cohen-Jones-Segal homeomorphism  $BC(f) \cong M$  is a diffeomorphism.

Our construction translates a cellular stratified space  $X$  into a topological acyclic category  $C(X)$ . When the stratification is totally normal, the face category has the discrete topology. When  $X$  is also regular,  $C(X)$  is a poset and can be studied by using Forman’s discrete Morse theory. It would become a useful tool, if we could extend discrete Morse theory to acyclic categories.

**Problem 5.10.** Extend discrete Morse theory to (topological) acyclic categories.

### 5.4 Toric Topology

As Examples 3.27 and 3.28 suggest, manifolds with torus actions seem to often be equipped with cellular decompositions satisfying our “niceness conditions”.

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<sup>30</sup>It is informed by Ernest Lupercio during the visit of the author to CINVESTAV in 2011 that there is a flaw in their argument. The author believes, however, that the main theorem in [CJS] should be true.

**Question 5.11.** Is it always true that the cell decomposition of a quasitoric manifold described by Davis and Januszkiewicz in §3 of their paper [DJ91] has a cylindrical structure? If so, what is the cylindrical face category? What about torus manifolds?

The description of the face category of  $\mathbb{CP}^n$  is particularly simple. As we have seen in Example 4.7, it is obtained from the Hasse diagram of the poset  $[n]$  by “labeling arrows by  $S^1$ ”. Compositions of morphisms are given by the group structure of  $S^1$ .

Notice that the poset  $[n]$  is the underlying poset of the acyclic category  $C(\mathbb{CP}^n)$ . It seems plausible that the cylindrical face category  $C(M)$  of a quasitoric manifold  $M$ , if it exists, can be obtained from the combinatorial structure of the simple polytope associated with  $M$ .

## A Simplicial Topology

In this appendix, we recall basic definitions and theorems in PL topology and simplicial homotopy theory used in this paper. Our main references are

- the book [RS72] by Rourke and Sanderson for PL topology, and
- the book [GJ99] by Goerss and Jardine for simplicial homotopy theory.

### A.1 Simplicial Complexes, Simplicial Sets, and Simplicial Spaces

Let us fix notations and terminologies for simplicial complexes first. Good references are the papers by Dwyer [DH01] and Friedman [Fri].

**Definition A.1.** For a set  $V$ , the power set of  $V$  is denoted by  $2^V$ .

**Definition A.2.** Let  $V$  be a set. An *abstract simplicial complex* on  $V$  is a family of subsets  $K \subset 2^V$  satisfying the following condition:

- $\sigma \in K$  and  $\tau \subset \sigma$  imply  $\tau \in K$ .

$K$  is called *finite* if  $V$  is a finite set.

**Definition A.3.** An *ordered simplicial complex*  $K$  is an abstract simplicial complex whose vertex set  $P$  is partially ordered in such a way that the induced ordering on each simplex is a total order.

An  $n$ -simplex  $\sigma \in K$  with vertices  $v_0 < \dots < v_n$  is denoted by  $\sigma = [v_0, \dots, v_n]$ .

There are several ways to define the geometric realization of an abstract simplicial complex.

**Definition A.4.** For an abstract simplicial complex  $K$  with vertex set  $V$ , define a space  $\|K\|$  by

$$\|K\| = \left\{ f \in \text{Map}(V, \mathbb{R}) \mid |\text{supp}(f)| < \infty, \sum_{v \in \sigma} f(v) = 1, f(v) \geq 0, \sigma \in K \right\},$$

where  $\text{supp}(f) = \{v \in V \mid f(v) \neq 0\}$  is the *support* of  $f : V \rightarrow \mathbb{R}$  and  $\text{Map}(V, \mathbb{R})$  is equipped with the compact-open topology.

The space  $\|K\|$  is called the *geometric realization* of  $K$ .

**Lemma A.5.** Suppose the vertex set  $V$  of an abstract simplicial complex  $K$  is finite. Choose an embedding

$$i : V \hookrightarrow \mathbb{R}^N$$

for a sufficiently large  $N$  so that the  $i(V)$  is affinely independent. Then we have a homeomorphism

$$\|K\| \cong \left\{ \sum_{v \in V} a_v i(v) \mid \sum_{v \in \sigma} a_v = 1, a_v \geq 0, \sigma \in K \right\}.$$

**Example A.6.** When  $V = \{0, \dots, n\}$ , consider the power set  $2^V$ . Then we have a homeomorphism

$$\|2^V\| \cong \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \right\} = \Delta^n.$$

The space  $\Delta^n$  is a convex polytope having  $(n+1)$  codimension-1 faces. Each codimension-1 face can be realized as the image of a map

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n$$

defined by

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

We also have maps

$$s^i : \Delta^n \longrightarrow \Delta^{n-1}$$

defined by

$$s^i(t_0, \dots, t_n) = (t_0, \dots, t_i + t_{i+1}, t_{i+2}, \dots, t_n).$$

□

For an ordered simplicial complex  $K$ , we may forget the ordering and apply the above construction. However, there is another construction.

**Definition A.7.** For an ordered simplicial complex  $K$  with vertex set  $V$ , let  $K_n$  be the set of  $n$ -simplices in  $K$ . Each element  $\sigma$  in  $K_n$  can be written as  $\sigma = (v_0, \dots, v_n)$  with  $v_0 < \dots < v_n$ . Under such an expression, define

$$d_i(\sigma) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

Define

$$\|K\| = \left( \coprod_{n=0}^{\infty} K_n \times \Delta^n \right) / \sim,$$

where the relation  $\sim$  is generated by

$$(\sigma, d^i(\mathbf{t})) \sim (d_i(\sigma), \mathbf{t}).$$

This is called the *geometric realization* of  $K$ .

**Lemma A.8.** For a finite ordered simplicial complex, the above two constructions of the geometric realization coincide.

The above construction can be extended to simplicial sets and simplicial spaces.

**Definition A.9.** A *simplicial set*  $X$  consists of

- a sequence of sets  $X_0, X_1, \dots$ ,
- a family of maps  $d_i : X_n \rightarrow X_{n-1}$  for  $0 \leq i \leq n$ ,
- a family of maps  $s_i : X_n \rightarrow X_{n+1}$  for  $0 \leq i \leq n$

satisfying the following relations

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i, & i < j \\ d_i \circ s_j = s_{j-1} \circ d_i, & i < j \\ d_j \circ s_j = 1 = d_{j+1} \circ s_j \\ d_i \circ s_j = s_j \circ d_{i-1}, & i > j + 1 \\ s_i \circ s_j = s_{j+1} \circ s_i, & i \leq j. \end{cases}$$

The maps  $d_i$  and  $s_j$  are called *face operators* and *degeneracy operators*, respectively.

When each  $X_n$  is a topological space and maps  $d_i, s_j$  are continuous,  $X$  is called a *simplicial space*.

**Remark A.10.** It is well known that giving a simplicial set  $X$  is equivalent to defining a functor

$$X : \Delta^{\text{op}} \longrightarrow \mathbf{Sets},$$

where  $\Delta$  is the full subcategory of the category of posets consisting of  $[n] = \{0 < 1 < \dots < n\}$  for  $n = 0, 1, 2, \dots$

**Example A.11.** For a topological space  $X$ , define

$$S_n(X) = \text{Map}(\Delta^n, X).$$

The operators  $d^i$  and  $s^i$  on  $\Delta^n$  induce

$$\begin{aligned} d_i & : S_n(X) \longrightarrow S_{n-1}(X) \\ s_i & : S_{n-1}(X) \longrightarrow S_n(X). \end{aligned}$$

When each  $S_n(X)$  is equipped with the compact-open topology, these maps are continuous and we obtain a simplicial space  $S(X)$ . This is called the *singular simplicial space*.

Usually  $S_n(X)$ 's are merely regarded as sets and  $S(X)$  is regarded as a simplicial set, in which case we denote it by  $S^\delta(X)$ .  $\square$

**Example A.12.** Let  $K$  be an ordered simplicial complex on the vertex set  $V$ . Define

$$s(K)_n = \left\{ \left[ \underbrace{v_0, \dots, v_0}_{i_0}, \underbrace{v_1, \dots, v_1}_{i_1}, \dots, \underbrace{v_k, \dots, v_k}_{i_k} \right] \mid \sigma = [v_0, \dots, v_k] \in K, \sum_{j=0}^k i_j = n \right\}.$$

Then the collection  $s(K) = \{s(K)_n\}$  becomes a simplicial set. This is called the *simplicial set generated by  $K$* .  $\square$

**Definition A.13.** The *geometric realization* of a simplicial space  $X$  is defined by

$$|X| = \left( \prod_{n=0}^{\infty} X_n \times \Delta^n \right) / \sim$$

where the relation  $\sim$  is generated by

$$\begin{aligned}(x, d^i(\mathbf{t})) &\sim (d_i(x), \mathbf{t}), \\ (x, s^i(\mathbf{t})) &\sim (s_i(x), \mathbf{t}).\end{aligned}$$

The map induced by the evaluation maps

$$S_n(X) \times \Delta^n \longrightarrow X$$

is denoted by

$$\text{ev} : |S(X)| \longrightarrow X.$$

Note that the geometric realization of an ordered simplicial complex is defined only by using face operators.

**Definition A.14.** A  $\Delta$ -set  $X$  consists of

- a sequence of sets  $X_0, X_1, \dots$ , and
- a family of maps  $d_i : X_n \rightarrow X_{n-1}$  for  $0 \leq i \leq n$ ,

satisfying the following relations

$$d_i \circ d_j = d_{j-1} \circ d_i,$$

for  $i < j$ .

When each  $X_n$  is equipped with a topology under which  $d_i$ 's are continuous,  $X$  is called a  $\Delta$ -space.

**Remark A.15.** The term  $\Delta$ -set is confusing, since a  $\Delta$ -set  $X$  is nothing but a functor

$$X : \Delta_{\text{inj}}^{\text{op}} \longrightarrow \mathbf{Sets},$$

where  $\Delta_{\text{inj}}$  is the subcategory of  $\Delta$  consisting of injective maps.

Thus a more appropriate terminology would be  $\Delta_{\text{inj}}$ -set. In this paper, however, we follow the terminology of Rourke and Sanderson [RS71], who initiated the study of homotopy-theoretic properties of such objects.

For functors taking values in other categories, we use the following terminology.

**Definition A.16.** For a category  $\mathbf{C}$ , a functor

$$X : \Delta_{\text{inj}}^{\text{op}} \longrightarrow \mathbf{C}$$

is called a  $\Delta_{\text{inj}}$ -object in  $\mathbf{C}$ .

**Definition A.17.** The *geometric realization* of a  $\Delta$ -space  $X$  is defined by

$$\|X\| = \left( \coprod_n X_n \times \Delta^n \right) / \sim,$$

where the relation  $\sim$  is generated by

$$(x, d^i(\mathbf{t})) \sim (d_i(x), \mathbf{t}).$$



**Remark A.18.** Note that any simplicial space  $X$  can be regarded as a  $\Delta$ -space. However the geometric realization of  $X$  as a  $\Delta$ -space,  $\|X\|$ , is much larger than that of  $X$  as a simplicial space.  $\|X\|$  is often called the fat realization.

In order to study the homotopy type of simplicial complexes, and more generally, regular cell complexes, the notion of regular neighborhood is useful. Let us recall the definition.

**Definition A.19.** Let  $K$  be a cell complex. For  $x \in K$ , define

$$\text{St}(x; K) = \bigcup_{x \in \bar{e}} e.$$

This is called the *open star* around  $x$  in  $K$ . For a subset  $A \subset K$ , define

$$\text{St}(A; K) = \bigcup_{x \in A} \text{St}(x; K).$$

When  $K$  is a simplicial complex and  $A$  is a subcomplex,  $\text{St}(A; K)$  is called the *regular neighborhood* of  $A$  in  $K$ .

The regular neighborhood of a subcomplex is often defined in terms of vertices.

**Lemma A.20.** *Let  $A$  be a subcomplex of a simplicial complex  $K$ . Then*

$$\text{St}(A; K) = \bigcup_{v \in \text{sk}_0(A)} \text{St}(v; K).$$

**Definition A.21.** Let  $K$  be a simplicial complex. We say a subcomplex  $L$  is a *full* subcomplex if, for any collection of vertices  $v_0, \dots, v_k$  in  $L$  which form a simplex  $\sigma$  in  $K$ , the simplex  $\sigma$  belongs to  $L$ .

The following fact is fundamental.

**Lemma A.22.** *If  $K$  is a simplicial complex and  $A$  is a full subcomplex, then  $A$  is a strong deformation retract of the regular neighborhood  $\text{St}(A; K)$ .*

*Proof.* The retraction

$$r_A : \text{St}(A; K) \longrightarrow A$$

is given by

$$r_A(x) = \frac{1}{\sum_{v \in A \cap \sigma} t(v)} \sum_{v \in A \cap \sigma} t(v)v,$$

if  $x = \sum_{v \in \sigma} t(v)v$  belongs to a simplex  $\sigma$ . A homotopy between  $i \circ r_A$  and the identity map is given by a “linear homotopy”. See Lemma 9.3 in [ES52], for more details.  $\square$

The following modification of this fact was proved in [BGRT].

**Lemma A.23.** *Let  $K$  be a simplicial complex and  $K'$  a subcomplex. Given a full subcomplex  $A$  of  $K$ , let  $A' = A \cap K'$ . Suppose we are given a strong deformation retraction  $H$  of  $\text{St}(A'; K')$  onto  $A'$ . Then there exists a deformation retraction  $\tilde{H}$  of  $\text{St}(A; K)$  onto  $A$  extending  $H$ .*

This Lemma is used to prove Lemma 4.18 in [BGRT]. In order to apply Lemma A.23 to prove Lemma 4.18, the following observation is crucial.

**Lemma A.24.** *Let  $K$  be a regular cell complex. For any stratified subspace  $L$  of  $K$ , the image of the regular neighborhood  $\text{St}(\text{Sd}(L); \text{Sd}(\overline{L}))$  of  $\text{Sd}(L)$  in  $\text{Sd}(\overline{L})$  under the embedding*

$$i : \text{Sd}(K) \hookrightarrow K$$

*contains  $L$ .*

*Proof.* For a point  $x \in L$ , there exists a cell  $e$  in  $L$  with  $x \in e$ . Under the barycentric subdivision of  $\overline{L}$ ,  $e$  is triangulated, namely there exists a sequence

$$\mathbf{e} : e_0 < e_1 < \cdots < e_n = e$$

of cells in  $\overline{L}$  such that

$$x \in i_{\mathbf{e}}(\text{Int}\Delta^n)$$

and

$$v(e) \in \overline{i_{\mathbf{e}}(\text{Int}\Delta^n)},$$

where  $v(e)$  is the vertex in  $\text{Sd}(\overline{L})$  corresponding to  $e$ . By definition of  $\text{St}$ , we have

$$i_{\mathbf{e}}(\text{Int}\Delta^n) \subset \text{St}(v(e); \text{Sd}(\overline{L})) = \text{St}(i(\text{Sd}(L)) \text{Sd}(\overline{L}))$$

and we have

$$L \subset \text{St}(i(\text{Sd}(L)); \text{Sd}(\overline{L})).$$

Conversely take an element  $y \in \text{St}(i(\text{Sd}(L)); \text{Sd}(\overline{L}))$ . There exists a simplex  $\sigma$  in  $\text{Sd}(\overline{L})$  and a point  $a \in \text{Sd}(L)$  with  $a \in \sigma$  and  $y \in \text{Int}(\sigma)$ .  $a$  can be taken to be a vertex. Thus there is a cell  $e$  in  $L$  with  $a = v(e)$ . By the definition of  $\text{Sd}(L)$ , there exists a chain

$$\mathbf{e} : e_0 < \cdots < e_n$$

in  $L$  containing  $e$  with  $\sigma = i_{\mathbf{e}}(\Delta^n)$ .

Since

$$\text{Int}(\sigma) \subset e_n \subset L,$$

we have  $y \in L$ . Thus we have proved

$$L = \text{St}(i(\text{Sd}(L)); \text{Sd}(\overline{L})).$$

It follows from the construction of the barycentric subdivision that  $i(\text{Sd}(L))$  is a full subcomplex of  $\overline{L}$ .  $\square$

*Proof of Lemma 4.18.* Let  $L = \text{Int}(D^n) \cup K$ . This is a stratified subspace of the regular cell decomposition on  $D^n$ . By Lemma A.24,  $L$  is a regular neighborhood of  $i(\text{Sd}(L))$  in  $\text{Sd}(\overline{L})$ . By Lemma A.22, there is a standard “linear” homotopy which contracts  $L$  on to  $i(\text{Sd}(L))$ .

By the construction of the homotopy, it can be taken to be an extension of a given homotopy on  $K$ , under the identification  $i(\text{Sd}(L)) = 0 * i(\text{Sd}(K))$ .  $\square$

## A.2 Locally Cone-like Spaces

Let us first define the operation  $*$  on subsets of a Euclidean space.

**Definition A.25.** For subspaces  $P, Q \subset \mathbb{R}^n$ , define

$$P * Q = \{(1-t)p + tq \mid p \in P, q \in Q, 0 \leq t \leq 1\}.$$

This is called the *convex sum* or *join* of  $P$  and  $Q$

When  $P$  is a single point  $v$ ,  $v * Q$  is called the *cone* on  $Q$  with vertex  $v$ .

**Remark A.26.** When  $P$  and  $Q$  are closed and “in general position”,  $P * Q$  agrees with the join operation.

**Definition A.27.** We say a subspace  $P \subset \mathbb{R}^n$  is *locally cone-like*, if, for any  $a \in P$ , there exists a compact subset  $L \subset P$  such that the cone  $a * L$  is a neighborhood of  $a$ .

**Remark A.28.** Locally cone-like spaces are called polyhedra in the Rourke-Sanderson book [RS72].

**Example A.29.** A vector (or affine) subspace of  $\mathbb{R}^n$  is locally cone-like. □

**Example A.30.** The half space  $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_n \geq 0\}$  is locally cone-like. More generally, given a finite collection of affine 1-forms  $L = \{\ell_1, \dots, \ell_k\}$  in  $\mathbb{R}^n$ , the intersection of closed half spaces defined by  $L$

$$P = \bigcap_{i=1}^k \{\mathbf{x} \in \mathbb{R}^n \mid \ell_i(\mathbf{x}) \geq 0\}$$

is locally cone-like. Such spaces are called  $\mathcal{H}$ -polyhedra in Ziegler’s book [Zie95].

When  $P$  is bounded, it is called an  $\mathcal{H}$ -polytope. It is a fundamental fact<sup>31</sup> that any  $\mathcal{H}$ -polytope can be written as a convex hull of a finite number of points

$$P = \text{Conv}(\mathbf{x}_1, \dots, \mathbf{x}_s) = \left\{ \sum_{i=1}^s \lambda_i \mathbf{x}_i \mid \sum_{i=1}^s \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

□

**Example A.31.** Given a finite collection of points  $\mathbf{x}_1, \dots, \mathbf{x}_s \in \mathbb{R}^n$ , define the *conical hull* by

$$\text{Cone}(\mathbf{x}_1, \dots, \mathbf{x}_s) = \left\{ \sum_{i=1}^s \lambda_i \mathbf{x}_i \mid \lambda_i \geq 0 \right\}.$$

Such a space is locally cone-like. In fact, it is a fundamental fact<sup>32</sup> that such a space can be written as an  $\mathcal{H}$ -polyhedra. More generally,  $P$  is an  $\mathcal{H}$ -polyhedron if and only if it can be written as the Minkowski sum of a conical hull and a convex hull, where the Minkowski sum  $A + B$  of subsets  $A, B \subset \mathbb{R}^n$  is given by

$$A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}.$$

□

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<sup>31</sup>Theorem 1.1 in [Zie95].

<sup>32</sup>Theorem 1.2 in [Zie95].

**Definition A.32.** When  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_s$  is affinely independent, the Minkowski sum

$$\{\mathbf{x}_0\} + \text{Cone}(\mathbf{x}_1, \dots, \mathbf{x}_s)$$

is called a *simplicial cone*.

**Lemma A.33.** *The class of locally cone-like spaces is closed under the following operations:*

- *finite intersections,*
- *finite products, and*
- *locally finite unions.*

**Corollary A.34.** *Euclidean polyhedral complexes are locally cone-like.*

The following theorem characterizes compact locally cone-like spaces.

**Theorem A.35.** *Any compact locally cone-like space can be expressed as a union of a finite number of simplices.*

*Proof.* See Theorem 2.11 in [RS72]. □

### A.3 PL Maps Between Polyhedral Complexes

In PL topology, we study triangulated spaces. Following Rourke and Sanderson, let us consider locally cone-like spaces. We also consider polyhedral complexes in a Euclidean space.

**Definition A.36.** Let  $P$  and  $Q$  be locally cone-like spaces. A map

$$f : P \longrightarrow Q$$

is said to be *piecewise-linear (PL)* if, for each  $a \in P$ , there exists a cone neighborhood  $N = a * L$  such that

$$f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$$

for  $x \in L$  and  $\lambda, \mu \geq 0$  and  $\lambda + \mu = 1$ .

**Lemma A.37.** *PL maps are closed under the following operations:*

- *products,*
- *compositions,*
- *the cone construction.*

Another important property of PL maps is extendability.

**Lemma A.38.** *Let  $P$  be a convex polytope and*

$$f : \text{Int}P \longrightarrow \mathbb{R}^n$$

*be a PL map. Then it has a PL extension*

$$\tilde{f} : P \longrightarrow \mathbb{R}^n.$$

**Theorem A.39.** *Let  $K$  and  $L$  be Euclidean polyhedral complexes. For any PL map*

$$f : K \longrightarrow L,$$

*there exist simplicial subdivisions  $K'$  and  $L'$  of  $K$  and  $L$ , respectively, such that the induced map*

$$f : K' \longrightarrow L'$$

*is simplicial.*

*Proof.* By Theorem 2.14 in [RS72]. □

## B Topological Categories

In this second appendix, we recall basics of topological categories. Our references are Segal's paper [Seg68] and the article by Dwyer in [DH01].

### B.1 Acyclic Topological Categories

**Definition B.1.** A *topological quiver*  $X$  is a diagram of spaces of the form

$$s, t : X_1 \longrightarrow X_0.$$

For a topological quiver  $X$  and  $n \geq 1$ , define

$$N_n(X) = \{(u_n, \dots, u_1) \in X_1^n \mid s(u_n) = t(u_{n-1}), \dots, s(u_2) = t(u_1)\}.$$

We also denote  $N_0(X) = X_0$ . An element of  $N_n(X)$  is called an  $n$ -chain of  $X$ .

**Definition B.2.** A *topological category*  $X$  is a topological quiver equipped with two more maps

$$\begin{aligned} i &: X_0 \longrightarrow X_1 \\ \circ &: N_2(X) \longrightarrow X_1 \end{aligned}$$

making the following diagrams commutative:

$$\begin{array}{ccccc} N_3(X) & \xrightarrow{\circ \times 1} & N_2(X) & & X_1 & \xrightarrow{(i \circ t) \times 1} & N_2(X) & \xleftarrow{1 \times (i \circ s)} & X_1 \\ \downarrow 1 \times \circ & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\ N_2(X) & \xrightarrow{\circ} & X_1, & & X_1. & & X_1. & & X_1. \end{array}$$

Elements of  $X_0$  are called *objects*. An element  $u \in X_1$  with  $s(u) = x$  and  $t(u) = y$  is called a *morphism* from  $x$  to  $y$  and is denoted by  $u : x \rightarrow y$ . The subspace of morphisms from  $x$  to  $y$  is denoted by  $X(x, y)$ , i.e.  $X(x, y) = s^{-1}(x) \cap t^{-1}(y)$ . For  $x \in X_0$ ,  $i(x)$  is called the *identity morphism* on  $x$  and is denoted by  $1_x : x \rightarrow x$ .

**Definition B.3.** A topological category is said to be *acyclic* if, for any pair of distinct objects  $x, y \in X_0$ , either  $X(x, y)$  or  $X(y, x)$  is empty and, for any object  $x \in X_0$ ,  $X(x, x)$  consists of the identity morphism.

**Definition B.4.** For a topological category  $X$ , define a relation  $\leq$  on  $X_0$  as follows:

$$x \leq y \text{ if and only if } X(x, y) \neq \emptyset.$$

**Lemma B.5.** When  $X$  is acyclic, the relation  $\leq$  is a partial order.

**Definition B.6.** For an acyclic topological category  $X$ , the poset  $(X_0, \leq)$  is called the *underlying poset* of  $X$  and is denoted by  $P(X)$ . The canonical functor from  $X$  to  $P(X)$  is denoted by

$$u_X : X \longrightarrow P(X).$$

**Remark B.7.** The reader might think that using  $P$  for underlying posets conflicts with the symbol  $P$  for the face poset of a stratified space in Definition 2.5.

As we have seen in Lemma 4.3, the face category  $F(X)$  of a cellular stratified space is an acyclic topological category and the underlying poset  $P(F(X))$  coincides with the face poset  $P(X)$  of  $X$ . The author chose to abuse the symbol  $P$  in order to emphasize this fact.

## B.2 Nerves and Classifying Spaces

One of the most fundamental facts on topological categories is that the collection  $N(X) = \{N_n(X)\}_{n \geq 0}$  defined in Definition B.1 forms a simplicial space.

**Lemma B.8.** *For a topological category  $X$ , we have*

$$N_n(X) = \text{Funct}([n], X),$$

where the poset  $[n] = \{0, \dots, n\}$  is regarded as a topological category (with discrete topology). Thus  $N(X)$  defines a functor

$$N(X) : \Delta^{\text{op}} \longrightarrow \mathbf{Spaces}.$$

In other words,  $N(X)$  is a simplicial space for any topological category  $X$ .

**Definition B.9.** The simplicial space  $N(X)$  is called the *nerve* of  $X$ . The source and target maps on  $X$  can be extended to

$$s, t : N_k(X) \longrightarrow X_0$$

by  $s(\mathbf{f}) = f(0)$  and  $t(\mathbf{f}) = f(k)$ , respectively. These are also called source and target maps.

**Definition B.10.** The geometric realization of  $N(X)$  is called the *classifying space* of  $X$  and is denoted by  $BX$ .

When we form the geometric realization of a simplicial space, nondegenerate chains are essential.

**Definition B.11.** For a topological category  $X$ , define

$$\overline{N}_n(X) = N_n(X) - \bigcup_i s_i(N_{n-1}(X)).$$

Elements of  $\overline{N}_n(X)$  are called *nondegenerate  $n$ -chains*.

**Lemma B.12.** *When  $P$  is a poset regarded as a small category,  $\overline{N}(P) = \{\overline{N}_n(P)\}$  is an ordered simplicial complex and we have*

$$BP = \|\overline{N}(P)\|.$$

More generally, we have the following description.

**Lemma B.13.** *When  $X$  is an acyclic topological category, the simplicial structure on  $N(X)$  can be restricted to give a structure of  $\Delta$ -space on  $\overline{N}(X)$ . Furthermore the composition*

$$\|\overline{N}(X)\| \longrightarrow \|N(X)\| \longrightarrow |N(X)| = BX$$

*is a homeomorphism.*

**Example B.14.** For any abstract or ordered simplicial complex  $K$ , the geometric realization  $\|K\|$  is a regular cell complex and the face category  $F(X) = C(X)$  is a poset. The structure of ordered simplicial complex on  $\overline{N}(C(X))$  coincides with the barycentric subdivision of  $K$ .  $\square$

## References

- [Abr00] Aaron Abrams. *Configuration Spaces and Braid Groups of Graphs*. PhD thesis, University of California at Berkeley, 2000, <http://www.math.uga.edu/~abrams/research/papers/thesis.ps>.
- [AGH] Aaron Abrams, David Gay, and Valerie Hower. Discretized configurations and partial partitions, arXiv:1009.2935.
- [AS94] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994, arXiv:hep-th/9304087.
- [BGRT] Ibai Basabe, Jesús González, Yuli B. Rudyak, and Dai Tamaki. Higher topological complexity and homotopy dimension of configuration spaces on spheres, arXiv:1009.1851.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [BZ92] Anders Björner and Günter M. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5(1):105–149, 1992.
- [CJS] R. L. Cohen, J.D.S. Jones, and G. B. Segal. Morse theory and classifying spaces, <http://math.stanford.edu/~ralph/morse.ps>. preprint.
- [CJS94] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Floer’s infinite-dimensional Morse theory and homotopy theory. *Sūrikaiseikikenkyūsho Kōkyūroku*, (883):68–96, 1994, <http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/0883-4.pdf>. Geometric aspects of infinite integrable systems (Japanese) (Kyoto, 1993).
- [CJS95] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Floer’s infinite-dimensional Morse theory and homotopy theory. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 297–325. Birkhäuser, Basel, 1995.
- [Dav78] Michael Davis. Smooth  $G$ -manifolds as collections of fiber bundles. *Pacific J. Math.*, 77(2):315–363, 1978.
- [Dav83] Michael W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Ann. of Math. (2)*, 117(2):293–324, 1983.
- [DCP95] C. De Concini and C. Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3):459–494, 1995.
- [DCS00] C. De Concini and M. Salvetti. Cohomology of Coxeter groups and Artin groups. *Math. Res. Lett.*, 7(2-3):213–232, 2000.
- [dD] Giacomo d’Antonio and Emanuele Delucchi. A Salvetti complex for toric arrangements and its fundamental group, arXiv:1101.4111.
- [DH01] William G. Dwyer and Hans-Werner Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics—CRM Barcelona. Birkhäuser Verlag, Basel, 2001.
- [dH08] Matias L. del Hoyo. On the subdivision of small categories. *Topology Appl.*, 155(11):1189–1200, 2008, arXiv:0707.1718.

- [DJ91] Michael W. Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62(2):417–451, 1991.
- [ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.
- [FM94] William Fulton and Robert MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.
- [For95] Robin Forman. A discrete Morse theory for cell complexes. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 112–125. Internat. Press, Cambridge, MA, 1995.
- [For98] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.
- [Fri] Greg Friedman. An elementary illustrated introduction to simplicial sets, arXiv:0809.4221.
- [FT] Mizuki Furuse and Dai Tamaki. On combinatorial models for configuration spaces of graphs. in preparation.
- [Ghr01] Robert Ghrist. Configuration spaces and braid groups on graphs in robotics. In *Knots, braids, and mapping class groups—papers dedicated to Joan S. Birman (New York, 1998)*, volume 24 of *AMS/IP Stud. Adv. Math.*, pages 29–40. Amer. Math. Soc., Providence, RI, 2001, arXiv:math.GT/9905023.
- [GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [HM03] Akio Hattori and Mikiya Masuda. Theory of multi-fans. *Osaka J. Math.*, 40(1):1–68, 2003, <http://projecteuclid.org/getRecord?id=euclid.ojm/1153493035>.
- [Kir] Alexander Kirillov, Jr. On piecewise linear cell decompositions, arXiv:1009.4227.
- [Kir88] Frances Kirwan. *An introduction to intersection homology theory*, volume 187 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988.
- [Koz08] Dmitry Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.
- [LW69] Albert T. Lundell and Stephen Weingram. *Topology of CW-Complexes*. Van Nostrand Reinhold, New York, 1969.
- [Mil56] John Milnor. Construction of universal bundles. I, II. *Ann. of Math. (2)*, 63:272–284, 430–436, 1956.
- [MS] Luca Moci and Simona Settepanella. The homotopy type of toric arrangements, arXiv:1009.3622.
- [Mun00] James R. Munkres. *Topology*. Prentice-Hall Inc., Englewood Cliffs, N.J., 2nd. edition, 2000.
- [Nog] Kazunori Noguchi. The Euler characteristics of categories and the barycentric subdivision, arXiv:1104.3630.



- [Nog11] Kazunori Noguchi. The Euler characteristic of acyclic categories. *Kyushu J. Math.*, 65(1):85–99, 2011, arXiv:1004.2547.
- [Pfl01] Markus J. Pflaum. *Analytic and geometric study of stratified spaces*, volume 1768 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
- [RS71] C. P. Rourke and B. J. Sanderson.  $\Delta$ -sets. I. Homotopy theory. *Quart. J. Math. Oxford Ser. (2)*, 22:321–338, 1971.
- [RS72] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer-Verlag, New York, 1972.
- [Sal87] Mario Salvetti. Topology of the complement of real hyperplanes in  $\mathbb{C}^N$ . *Invent. Math.*, 88(3):603–618, 1987.
- [Sch03] Jörg Schürmann. *Topology of singular spaces and constructible sheaves*, volume 63 of *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)]*. Birkhäuser Verlag, Basel, 2003.
- [Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, 34:105–112, 1968.
- [Tama] Dai Tamaki. Cellular Stratified Spaces II: Basic Constructions, arXiv:1111.4774.
- [Tamb] Dai Tamaki. The Grothendieck Construction and Gradings for Enriched Categories, arXiv:0907.0061.
- [Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.